Please attempt the problems on your own before consulting the answer key!

Small note on this answer key: I started using footnotes which look like this[0]. They are there whenever there is more to something that I don't want to cover within the text because the information is not entirely relevant/necessary, or when I have some extra resources.

1. **Let N be a positive integer with exactly 4 distinct prime factors, each of which is raised to the first power. How many positive integers divide N without remainder?**

Answer: 16

Solution: Before I begin, if you notice (like Avaneesh did) that the answer is just 2^4, great job! If not, let's solve it as if we didn't know that pattern.

So, we have a positive integer with four distinct factors (which is the same as saying four prime factors). Let's say we have four prime factors a, b, c, d and $a * b * c * d = N$ (if you want to assign actual numbers, feel free to assign the smallest possible prime numbers being $2, 3, 5, 7$, $N = abcd = 210$). Now let's start listing factors of N (factors of N are positive integers that divide N without remainder):

- 1, $abcd$ | 1 and the integer itself ($N = abcd$) are distinct factors
- a, b, c, d | the prime factors are distinct factors
- ab, ac, ad, bc, bd, cd | all combinations of two factors are distinct factors
- abc, abd, acd, adc | all combinations of three factors are distinct factors

Counting all the factors we found we get:

$$2 + 4 + 6 + 4 = 16$$

Therefore, there are 16 positive integers that divide N without remainder[1]. Thus, our answer is 16.

2. **Alice and Bob play the following game. Laying on the table are 26 stones. The players alternatively remove 1, 2, 3, 4, or 5 stones from the table, with Alice going first. Whoever takes the last stone wins. There is one value of k such that Alice can guarantee a win by taking k stones on her first turn and then playing perfectly. What is the value of k?**

Answer: 2

Solution: Awesome question here! This one requires a bit of creative thinking. Let's get into it.

First, let's lay out how the game works. There are two players and a pile of 26 stones. Turn-by-turn, they each take anywhere from 1 to 5 stones from the pile, and Alice goes first. The person that takes the last stones wins. Now, we are told that there is a move that Alice can make on her first turn to guarantee that she wins (if she plays perfectly afterwards). Looking at the possible moves, notice that whatever move a player makes, it is always possible for the other player to play their move such that the stones removed across the two turns is 6. For example, if player one removes 2 stones, player two can remove 4 stones ($2 + 4 = 6$) and if player one removes 5 stones, player two can remove 1 stone ($5 + 1 = 6$). Thus, player two is able to keep the number of stones at a multiple of 6 from the original amount. So, Alice's best move is to take a certain number of stones such that the number of stones remaining is a multiple of 6 so that no matter what Bob plays, she can keep removing the complementary number of stones such that 6 stones are removed every round. To lower the number of stones on the table to a multiple of 6, she can remove $k = 2$ stones on her first turn to bring the number of stones down to 24. Now, whatever move Bob plays (for example, 1 stone removed), she can always remove the complementary number of stones (continuing the example, it would be 5 stones removed). Alice and Bob

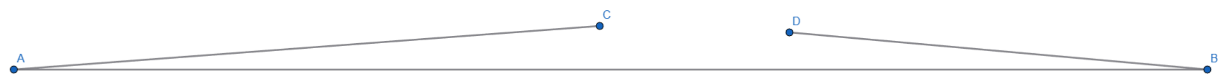
[1] Note that because each of the prime factors are raised to the power of 1, there are only 16 distinct factors. Had this not been the case, there would be infinitely many factors depending on what power each of the distinct prime factors is raised to (Try and figure out why!).

would keep removing 6 stones every round until there are 0 stones remaining, with Alice making the last move and thereby winning the game. Thus, our answer is 2.

3. **Real numbers a and b are chosen with $1 < a < b$ such that there are no triangles with positive area whose sides have lengths $1, a,$ and b or $\frac{1}{a}, \frac{1}{b},$ and 1. What is the smallest possible value of b?**

Answer: $\frac{3+\sqrt{5}}{2}$

Solution: This question is algebra heavy with geometry principles so it may be helpful for those who have a strong algebra background and want to begin with geometry. Moving on, let's establish some background information. In order for a triangle to, well, be a triangle, the sum of its legs (small sides) must be greater than its hypotenuse[2] (long side). For example, view the image below:



Because the total length of the would-be triangle's small sides, $\overline{AC}$ and $\overline{BD}$, are less than the length of the hypotenuse, $\overline{AB}$, the figure cannot be a triangle. In order for a triangle to exist, it must satisfy $Leg_1 + Leg_2 > Hypotenuse$.

With that in mind, let's get back to the question at hand. We have two real numbers a and b which satisfy $1 < a < b$ and it is not possible to create a triangle with side lengths $1, a,$ and b. If it was possible to make a triangle with side lengths $1, a,$ and b, then the sum of the two legs, 1 and a, would have to be greater than the hypotenuse, b: $1 + a > b$. Since it is not possible to make a triangle, the opposite of the equation must be true: $1 + a \leq b$. Moving on, we also know that a triangle with side lengths $\frac{1}{a}, \frac{1}{b},$ and 1 is also not possible. Note that because $1 < a < b$, $1 > \frac{1}{a} > \frac{1}{b}$ is true, so 1 is the hypotenuse, and $\frac{1}{a}$ and $\frac{1}{b}$ are the legs. Anyways, if it were possible to create a triangle then $\frac{1}{a} + \frac{1}{b} > 1$ would hold true. Since it's not possible, the opposite is true: $\frac{1}{a} + \frac{1}{b} \leq 1$. Let's write this in an easier to interpret form:

- $\frac{1}{a} + \frac{1}{b} \leq 1$ | original equation
- $b + a \leq ab$ | multiplying both sides by ab
- $b \leq ab - a$ | bringing a terms to one side
  - Bringing b terms to one side is totally acceptable as well, just note that you may have to solve for b instead of a in future equations
- $b \leq a(b - 1)$ | factoring
- $\frac{b}{b-1} \leq a$ | isolating a

Since we solved for a in this equation, let's also solve for a in our earlier equation $1 + a \leq b$:

- $1 + a \leq b$ | original equation
- $a \leq b - 1$ | isolating a

Great, we now have two equations that we know must hold true. Combining them into one equation we have:

$$\frac{b}{b-1} \leq a \leq b - 1$$

[2] Technically the sum of the lengths of the legs can equal the hypotenuse, but then the triangle will have 0 area (visually, it will appear as a line). Although I don't state it, I abstract that triangles must have a positive area (the question likewise mandates it). Note that 0 is not positive.

In order for an a value to exist, the right side, $b - 1$, must be equal to or greater than the left side, $\frac{b}{b-1}$ (otherwise there would be no range of values for a). Note that we are told $b > 1$. The graph of $\frac{b}{b-1}$ has a vertical asymptote at $b = 1$, and values to the right of it start at ∞ and rapidly decreases. On the other hand, $b - 1$ starts at 0 and increases at a constant rate. Therefore, $b - 1$ starts out less than $\frac{b}{b-1}$ for values past $b = 1$, so in order for $b - 1 \geq \frac{b}{b-1}$, there must be some point where the two functions intersect and the function $b - 1$ changes from being below to above $\frac{b}{b-1}$. Let's solve for this intersection point by setting both functions equal to each other:

- $b - 1 = \frac{b}{b-1}$ | original equation
- $(b - 1)^2 = b$ | multiplying both sides by b
- $b^2 - 3b + 1 = 0$ | expanding and all terms to the left side
- $b = \frac{-(-3) \pm \sqrt{3^2 - 4(1)(1)}}{2(1)}$ | using the quadratic formula with $a = 1, b =- 3, c = 1$
  - Using the quadratic formula because factoring did not work
- $b = \frac{3 \pm \sqrt{5}}{2}$ | simplifying
- $b = \frac{3 + \sqrt{5}}{2}$ | disregarding the solution that is ≤ 1

This is the first value of b where $b - 1 \geq \frac{b}{b-1}$ (and $b > 1$), and the first value where a has a possible value. Therefore, the smallest possible value of b is $b = \frac{3+\sqrt{5}}{2}$. Thus, our answer is $\frac{3+\sqrt{5}}{2}$.

4. **Let $\lfloor x \rfloor$ be the greatest integer less that or equal to x. For $0 < x \leq 100$, the values of x satisfying $\lfloor log_{10}(x + 2) \rfloor = \lfloor log_{10}x \rfloor + 2$ can be written as $a \leq x < b$. Find a and b. Present as an ordered pair (a, b).**

Answer: $(\frac{1}{100}, \frac{1}{10})$

Solution: First of all, make sure you understand the basic premise of flooring[3] and logarithms (which are covered in algebra 2). Moving on, let's try and find the values of x that satisfy the function. Personally, I recommend testing values. First, let's try any value ≥ 1. Plugging in 1 directly, we get:

- $\lfloor log_{10}(x + 2) \rfloor = \lfloor log_{10}x \rfloor + 2$ | original equation
- $\lfloor log_{10}(1 + 2) \rfloor = \lfloor log_{10}1 \rfloor + 2$ | plugging in $x = 1$
- $\lfloor log_{10}(3) \rfloor = \lfloor 0 \rfloor + 2$ | substituting $log_{10}1 = 0$
- $0 = 0 + 2$ | substituting $\lfloor log_{10}(3) \rfloor = 0$ because $0 < log_{10}3 < 1$
- $0 \neq 2$ | simplifying

So, $x = 1$ is not valid. Looking at the equation, inputting any values > 1 will not work because $\lfloor log_{10}(x + 2) \rfloor$ will never be 2 more than $\lfloor log_{10}x \rfloor$ (the most possible is being greater by 1, which occurs, for example, when $x = 8$). Therefore, we must look at values $0 < x < 1$. Looking at the equation, for all possible values $0 < x < 1$, $\lfloor log_{10}(x + 2) \rfloor$ will always be $= 0$. Therefore, our equation is now:

$$0 = \lfloor log_{10}x \rfloor + 2 \text{ or } \lfloor log_{10}x \rfloor =- 2$$

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[3] Flooring is converting a decimal into the greatest integer equal to or less than the decimal. It is represented by ⌊x⌋, where x is the decimal to be floored. Below are some examples:
⌊3.3⌋ = 3; ⌊3.7⌋ = 3; ⌊4⌋ = 4; ⌊127.4⌋ = 127; ⌊0.4⌋ = 0.

The first value for which $log_{10}x = -2$ is $x = 10^{-2} = \frac{1}{100}$. Since $log_{10}x$ is floored, any value up till $log_{10}x = -1$ will equal -2. $log_{10}x = -1$ at $x = 10^{-1} = \frac{1}{10}$. Thus, the domain[4] for our function is $\frac{1}{100} \leq x < \frac{1}{10}$. Since the question requests our answer in the form (a, b) where $a \leq x < b$, our final answer is $(\frac{1}{100}, \frac{1}{10})$.

5. **In quadrilateral $MNPQ$, two non-intersecting circles are located in such a way that one circle touches $\overline{MN}$, $\overline{NP}$, and $\overline{PQ}$, and the other circle touches $\overline{MN}$, $\overline{MQ}$, and $\overline{PQ}$. Points B and A lie on $\overline{MN}$ and $\overline{PQ}$ respectively, and $\overline{AB}$ is tangent to both circles. Compute $\overline{MQ}$ in terms of b and p, where $b = \overline{NP}$, and the perimeter of quadrilateral $BAQM$ is greater than the perimeter of quadrilateral $BAPN$ by $2p$. Express your answer in terms of only b and p.**

Answer: $p + b$

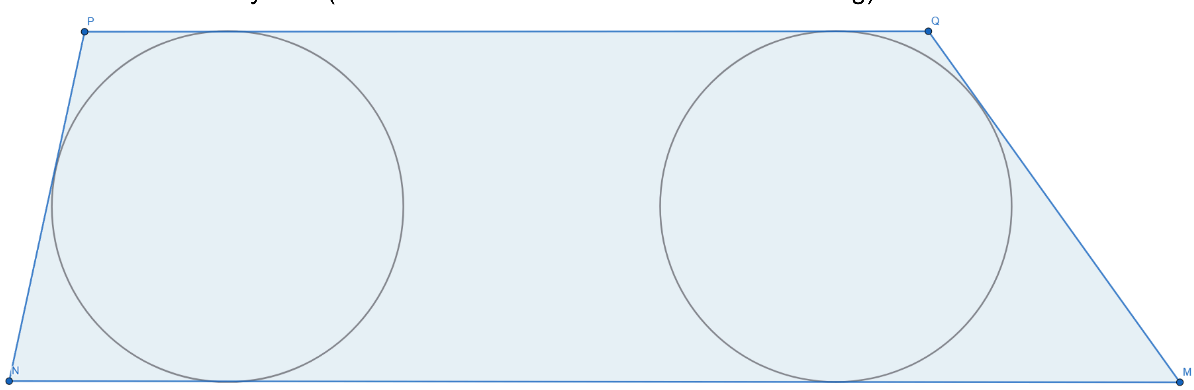
Solution: First of all, for those who weren't able to find the solution on their own, before reading through the explanation please view the following screenshot from the Wikipedia[5]:

According to the Pitot theorem, the two pairs of opposite sides in a tangential quadrilateral add up to the same total length, which equals the semiperimeter s of the quadrilateral:

$$a + c = b + d = \frac{a + b + c + d}{2} = s.$$

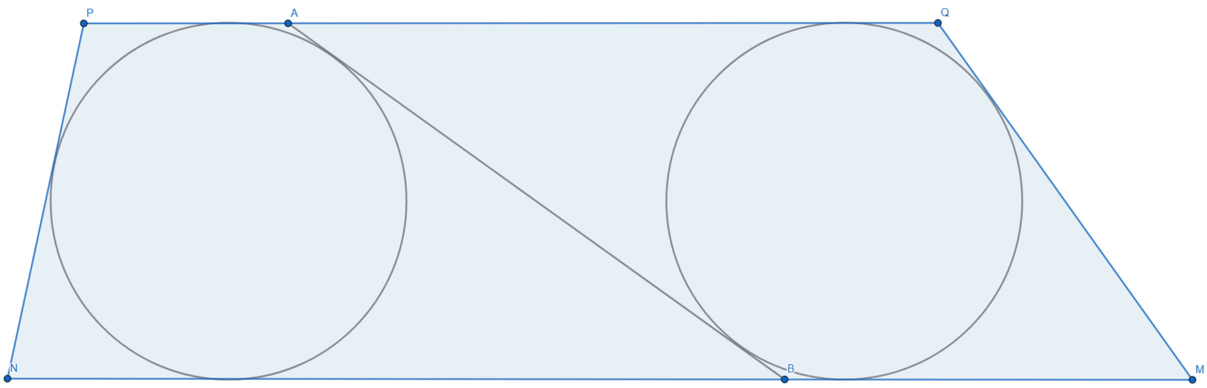
If you weren't able to solve the problem, please use this information and try again!

Moving on, let's first create a diagram. We have a quadrilateral $MNPQ$, and it contains two circles, one of which is tangential[6] to $\overline{MN}$, $\overline{NP}$, and $\overline{PQ}$, and the other which is tangential to $\overline{MN}$, $\overline{MQ}$, and $\overline{PQ}$. Let's create a sketch of how this may look (also note that the circles are non-intersecting):



Next, the question tells that there are two points, B, and A, that lie on $\overline{MN}$ and $\overline{PQ}$ respectively in such a way that $\overline{AB}$ is tangent to both circles. Let's construct that (changing our diagram as needed):

[4] The set of all possible x values
[5] https://en.wikipedia.org/wiki/Tangential_quadrilateral#Characterizations. I actually wasn't able to solve the problem originally, but while reading through the Wiki I found this and was able to solve it lol. However, because of how overcomplicated the math topics on the Wiki often are, I wouldn't really recommend using it.
[6] Tangential means to touch at only one point



Awesome, we've satisfied all the conditions of the question. Now let's get to solving. We are asked to solve for $\overline{MQ}$ in terms of b and p, where $b = \overline{NP}$ and the perimeter of quadrilateral $BAQM$ is greater than the perimeter of quadrilateral $BAPN$ by $2p$. The perimeter of a quadrilateral is the sum of its four sides, so we can make the equation:

$$\overline{BA} + \overline{AQ} + \overline{QM} + \overline{MB} = \overline{BA} + \overline{AP} + \overline{PN} + \overline{NB} + 2p$$

Now, according to the Pitot Theorem, in a tangential quadrilateral the sums of opposite sides are equal. Notice that both quadrilaterals $BAQM$ and $BAPN$ are tangential quadrilaterals because both of them have a circle inscribed within them that is tangent to each side. Therefore, the sum of opposite sides in these equilaterals are equal. Thus in tangential quadrilateral $BAQM$:

$$\overline{BA} + \overline{QM} = \overline{AQ} + \overline{MB}$$

And in tangential quadrilateral $BAPN$:

$$\overline{BA} + \overline{PN} = \overline{AP} + \overline{NB}$$

Notice that we can substitute both equations back into the original equation. Since both equations contain $\overline{BA}$, and one of them contains $\overline{QM}$, which we are solving for, I'll attempt to input them into the original equation (in hopes of cancelling out $\overline{BA}$ later and solving for $\overline{QM}$):

- $\overline{BA} + \overline{AQ} + \overline{QM} + \overline{MB} = \overline{BA} + \overline{AP} + \overline{PN} + \overline{NB} + 2p$ | original equation
- $\overline{BA} + \overline{QM} + \overline{AQ} + \overline{MB} = \overline{BA} + \overline{PN} + \overline{AP} + \overline{NB} + 2p$ | rearranging terms
- $\overline{BA} + \overline{QM} + \overline{BA} + \overline{QM} = \overline{BA} + \overline{PN} + \overline{AP} + \overline{NB} + 2p$ | substituting $\overline{BA} + \overline{QM} = \overline{AQ} + \overline{MB}$
- $\overline{BA} + \overline{QM} + \overline{BA} + \overline{QM} = \overline{BA} + \overline{PN} + \overline{BA} + \overline{PN} + 2p$ | substituting $\overline{BA} + \overline{PN} = \overline{AP} + \overline{NB}$
- $2(\overline{BA} + \overline{QM}) - 2(\overline{BA} + \overline{PN}) = 2p$ | regrouping and isolating $2p$
- $\overline{BA} + \overline{QM} - \overline{BA} - \overline{PN} = p$ | dividing both sides by 2 and expanding
  - Remember to distribute the -1
- $\overline{QM} = p + \overline{PN}$ | simplifying and isolating $\overline{QM}$
- $\overline{QM} = p + b$ | substituting $b = \overline{NP}$

Wow, what originally seemed like an algebraic nightmare simplified nicely. $\overline{QM}$ in terms of p and b is $p + b$. Thus, our answer is $p + b$.

6. **The equation x ln 27 log$_{13}$ $e = 27$log$_{13}$ y has integer solutions (x, y). Find the integer solution with the smallest value of y that exceeds 70 and present your answer as $x + y$.**

Answer: 117

Solution: This question tests your log knowledge. Let's do some algebraic shenanigans (aka algebraic manipulation)!

- $xln(27)log_{13}(e) = 27log_{13}(y)$ | original equation (with parenthesis for clarity)

- $xln(3^3)log_{13}(e) = 27log_{13}(y)$ | I am trying to simplify the equation using $log_a b^c = clog_a b$
- $3xln(3)log_{13}(e) = 27log_{13}(y)$ | using the prior mention property of logarithms
- $xln(3)log_{13}(e) = 9log_{13}(y)$ | dividing both sides by 3
- $xln(3)\frac{ln(e)}{ln(13)} = 9\frac{ln(y)}{ln(13)}$ | using the property $log_a b = \frac{log_c b}{log_c a}$
    - I used $c = e$ because $log_e(e) = 1$
    - Also note that log_e is the same as ln
- $xln(3) = 9ln(y)$ | multiplying both sides by $ln13$ and simplifying $ln(e) = 1$

Now let's think about this intuitively. x and y must be integers, and y must be the smallest possible value that is > 70. Therefore, if we have $ln(3)$ on the left side, we also need $ln(3)$ on the right side. Since y must be > 70, we have to use the smallest power of 3 that is greater than 70 so that we can have $ln(3)$ on the right side. That power is $3^4 = 81$. Thus, if we have $y = 3^4$, then:

- $xln(3) = 9ln(y)$ | original equation
- $xln(3) = 9ln(3^4)$ | plugging in $y = 3^4$
- $xln(3) = 9 * 4ln(3)$ | using the property $log_a b^c = clog_a b$
- $x = 36$ | dividing both sides by $ln(3)$ and simplifying $9 * 4$

Thus, solution to the equation that has the smallest possible value of y that is greater than 70 is the ordered pair $(36, 81)$. The question requests our answer in the form $x + y$, so: $x + y = 36 + 81 = 117$. Our answer is 117.

7. **Aaron took 30 numbers, rounded each of them up to the nearest integer, added the results, and got 2026. When he took the original numbers, rounded each of them down to the nearest integer and added the results, he got 2010. How many of his original numbers were integers?**

Answer: 14

Solution: First of all, I apologize for getting the wrong answer during the competition. I utilized a single test case that was flawed. I shouldn't have done that.

Moving on, let's get to solving this problem. Aaron has 30 numbers, and when they are rounded up to the nearest integer they sum to 2026. On the other hand, when they are rounded down to the nearest integer (aka floored) they sum to 2010. Now, think about this intuitively. When the integers were rounded up, they summed to 2026. However, when they were rounded down they summed to 2010, which is 16 less than 2026. Thus, 16 numbers must have changed dropped in value by 1. Only numbers that are not integers would change depending on whether they are rounded up or down. Thus, those 16 numbers must have had non-integer values. The other 14 numbers, whose values didn't change, must therefore be integers. Our answer is 14.

If you want to ensure you are correct, let's use a smaller test case. Let's take five numbers being $3, 3.5, 4.2, 6, 6.4$. The sum of each number rounded up is $3 + 4 + 5 + 6 + 7 = 25$. The sum of the floors of the numbers is $3 + 3 + 4 + 6 + 6 = 22$. By our previous discretion, because the sum changed by 3, there must be 3 non-integers the leftover 2 numbers must be integers. That is indeed the case.

8. **Solve $1 + 2^{\tan x} = 3 \cdot 4^{(\frac{sin(\frac{\pi}{4}-x)}{\sqrt{2}cosx})}$ where $0 < x \leq 2\pi$.**

Answer: $\frac{\pi}{4}, \frac{5\pi}{4}$

Solution: Yes, you can just plug in all the special values on the interval $0 < x \leq 2\pi$. Would I recommend it? Yes and no, because it may be faster than actually solving it, however if the answer isn't just standard unit circle values (for example if double angles are involved), then you may not find all the solutions. Therefore, I would encourage you to attempt to manually solve it but if you are stuck then just plug in values. Just make sure to solve it manually after the competition if you do decide to do that. With that out of the way, let's go ahead and solve it now.

So, first of all, $\frac{sin(\frac{\pi}{4}-x)}{\sqrt{2}cosx}$ is absolutely disgusting. Let's try and do something about it (simplify!):

- $\frac{sin(\frac{\pi}{4}-x)}{\sqrt{2}cosx}$ | original expression
- $\frac{sin(\frac{\pi}{4})cos(x) - cos(\frac{\pi}{4})sin(x)}{\sqrt{2}cosx}$ | sine subtraction formula: $sin(A - B) = sin(A)cos(B) - cos(A)sin(B)$
    - You can also technically use the addition formula (which I actually did) in case you forget the subtraction formula. You will get the same final answer.
- $\frac{\frac{\sqrt{2}}{2}cos(x) - \frac{\sqrt{2}}{2}sin(x)}{\sqrt{2}cosx}$ | evaluating $sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$ and $cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$
- $\frac{cos(x) - sin(x)}{2cosx}$ | dividing out $\sqrt{2}$ and moving the $\frac{1}{2}$ into the denominator as 2
    - Same thing as multiplying top and bottom by $\frac{2}{\sqrt{2}}$
- $\frac{1}{2}(1 - tan(x))$ | factoring out $\frac{1}{2}$ and dividing numerator by $cosx$

Nice, $\frac{1}{2}(1 - tan(x))$ is a lot easier to work with! Let's plug that back into the original equation and continue simplifying:

- $1 + 2^{\tan x} = 3 \cdot 4^{(\frac{sin(\frac{\pi}{4}-x)}{\sqrt{2}cosx})}$ | original equation
- $1 + 2^{\tan x} = 3 \cdot 4^{\frac{1}{2}(1 - tan(x))}$ | substituting $\frac{sin(\frac{\pi}{4}-x)}{\sqrt{2}cosx} = \frac{1}{2}(1 - tan(x))$
- $1 + 2^{tan(x)} = 3 * {4^{\frac{1}{2}}}^{(1-tan(x))}$ | properties of exponents $a^{b*c} = {a^{b}}^{c}$
- $1 + 2^{tan(x)} = 3 * 2^{(1-tan(x))}$ | $4^{\frac{1}{2}} = \sqrt{4} = 2$
- $1 + 2^{tan(x)} = 3 * 2^{1} * 2^{-tan(x)}$ | properties of exponents $a^{b+c} = a^{b} * a^{c}$
- $1 + 2^{tan(x)} = 6 * 2^{-tan(x)}$ | simplifying

Now, this is actually where I got stuck because there doesn't seem to be any more algebraic manipulation possible. However, if we let $y = 2^{tan(x)}$ it's easy to see that isn't the case:

- $1 + 2^{tan(x)} = 6 * 2^{-tan(x)}$ | original equation
- $1 + y = 6 * y^{-1}$ | substituting $y = 2^{tan(x)}$
- $y + y^{2} = 6$ | multiplying both sides by y
- $y^{2} + y - 6 = 0$ | bring terms to one side
- $(y + 3)(y - 2) = 0$ | factoring

Thus, y, which is $2^{tan(x)}$, must be equal to either -3 or 2. Let's solve for both scenarios:

- $y = -3$ | original equation
- $2^{tan(x)} = -3$ | substituting $y = 2^{tan(x)}$

It is not possible for a real exponent to turn a positive number negative. Therefore, $y = -3$ is not possible. Let's check $y = 2$:

- $y = 2$ | original equation

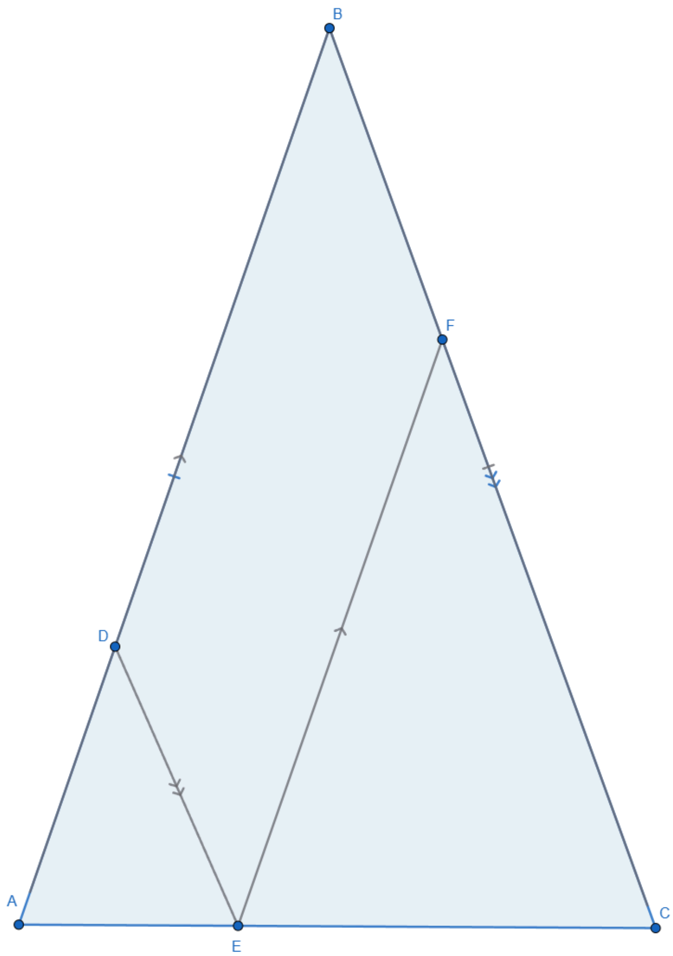
- $2^{tan(x)} = 2^{1}$ | substituting $y = 2^{tan(x)}$
- $tan(x) = 1$ | if the bases are the same, the exponents must be equal
- $x = \frac{\pi}{4}, \frac{5\pi}{4}$ | on the interval $0 < x \leq 2\pi$, $tan(x) = 1$ at $\frac{\pi}{4}$ and $\frac{5\pi}{4}$

Just in case, make sure to plug both values back into the original equation to make sure they work. They both work. Thus, our answer is $\frac{\pi}{4}, \frac{5\pi}{4}$.

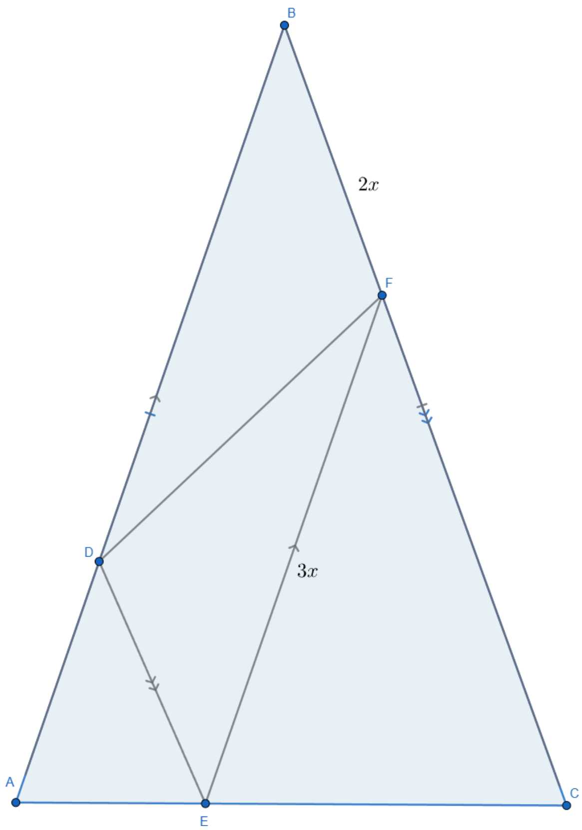
9. **In isosceles triangle ABC, point E is located on the base $\overline{AC}$, and points D and F are located on the lateral sides $\overline{AB}$ and $\overline{BC}$, respectively. $\overline{DE}$ is parallel to $\overline{BC}$ and $\overline{EF}$ is parallel to $\overline{AB}$. What part of the area of triangle ABC is the area of triangle DEF if $BF : EF = 2 : 3$?**

Answer: $\frac{6}{25}$

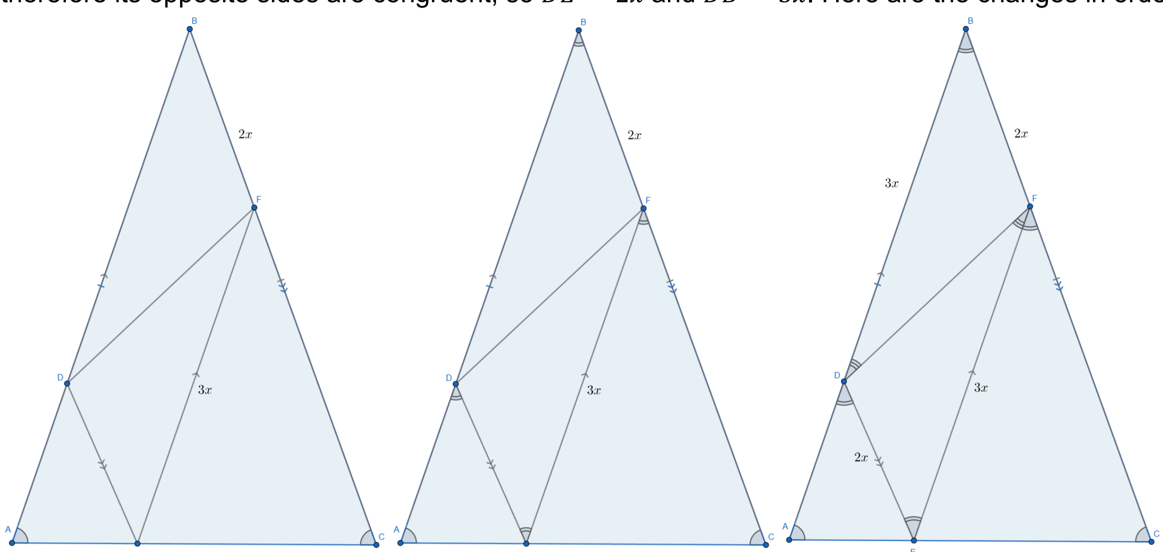
Solution: Geometry question! Diagram time! We have an isosceles triangle ABC with point E on the base $\overline{AC}$ (the side that isn't congruent to the other two sides). Additionally, points D and F are on $\overline{AB}$ and $\overline{BC}$ respectively in such a way that $\overline{DE}$ is parallel to $\overline{BC}$ and $\overline{EF}$ is parallel to $\overline{AB}$. Here's a potential diagram:



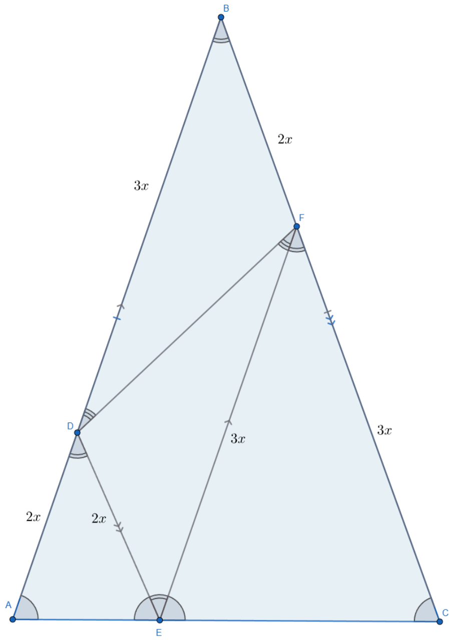
Nice, now we have a diagram to work with. We can always alter later if the need arises. Anyways, we are trying to solve for the ratio of $A_{\Delta ABC} : A_{\Delta DEF}$ and are told that $BF : EF = 2 : 3$. Let's draw $\overline{DF}$ and let $\overline{BF} = 2x$ and $\overline{EF} = 3x$:



Now, let's start solving. To begin with, since ΔABC is isosceles with congruent segments $\overline{AB}$ and $\overline{BC}$, the angles opposite the congruent sides, $\angle BAC$ and $\angle BCA$, are congruent. Additionally, due to the parallel sides, $\angle FED$ and $\angle EFC$ are alternate interior angles, and therefore congruent. On the same note, $\angle FED$ and $\angle ADE$ are congruent as they are also alternate interior angles. This makes all three angles congruent. Additionally, $\angle ADE$ and $\angle ABC$ are congruent as they are corresponding angles. This now makes four angles congruent. $\angle BDF$ and $\angle DFE$ are also congruent as they are alternate interior angles. Finally, notice that quadrilateral $DBFE$ has opposite sides that are parallel, meaning it is a parallelogram and that therefore its opposite sides are congruent, so $\overline{DE} = 2x$ and $\overline{DB} = 3x$. Here are the changes in order:



Now, because ΔABC has two angles that are single arcs and one angle that is a double arc. Since triangles ADE and EFC both have a single arc angle and double arc angle, the final angle must be a single arc angle because it is ΔABC's final angle. After marking that, we can see that triangles ADE and EFC are both isosceles (both have two single arc angles) and the two sides opposite the congruent angles will be congruent:



Now, we could continue, but we already have enough information. To begin with, $\Delta ADE \sim \Delta ABC$ by AA similarity and because the ratio of the side lengths is $\frac{\overline{DE}}{\overline{BC}} = \frac{2x}{5x} = \frac{2}{5}$, the ratio of the areas is $(\frac{2}{5})^2 = \frac{4}{25}$ (this means triangle ADE has $\frac{4}{25}$ the area of triangle ABC). Additionally, $\Delta EFC \sim \Delta ABC$ by AA similarity and with the ratio of the side lengths being $\frac{\overline{FC}}{\overline{BC}} = \frac{3x}{5x} = \frac{3}{5}$, the ratio of the areas is $(\frac{3}{5})^2 = \frac{9}{25}$. Now, notice that ΔDEF is congruent to ΔFBD by SAS congruence. Therefore, the areas of the two triangles are equal. With that final piece of information, we can solve for the area of ΔDEF in terms of ΔABC:

- $A_{\Delta ABC} = A_{\Delta ABC}$ | original equation
- $A_{\Delta ABC} = A_{\Delta ADE} + A_{\Delta EFC} + A_{\Delta DEF} + A_{\Delta FBD}$ | the sum of the triangle is the sum of its interior
- $A_{\Delta ABC} = \frac{4}{25}A_{\Delta ABC} + \frac{9}{25}A_{\Delta ABC} + A_{\Delta DEF} + A_{\Delta FBD}$ | substituting $A_{\Delta ADE} = \frac{4}{25}A_{\Delta ABC}$ and $A_{\Delta EFC} = \frac{9}{25}A_{\Delta ABC}$
- $\frac{12}{25}A_{\Delta ABC} = A_{\Delta DEF} + A_{\Delta FBD}$ | simplifying
- $\frac{12}{25}A_{\Delta ABC} = A_{\Delta DEF} + A_{\Delta DEF}$ | triangles ΔDEF and ΔFBD are congruent
- $\frac{12}{25}A_{\Delta ABC} = 2(A_{\Delta DEF})$ | combining like terms
- $\frac{6}{25}A_{\Delta ABC} = A_{\Delta DEF}$ | dividing both sides by 2

Therefore, the area of triangle DEF is $\frac{6}{25}$ the area of triangle ABC. Our answer is $\frac{6}{25}$.

10. Find real a such that the solution of the following system is an interval of length 5:

$$\begin{cases} x^2 - x(3a-2) + 2a^2 - 4a < 0 \\ x < 3 \end{cases}$$

Answer: -7

Solution: Top tier question! Great test to see if you understand systems of equations. Moving on, we want to find a real a such that the solution to the system of equations has an interval of 5. The second equation is just there to tell us that the interval of length 5 must occur before $x = 3$. The quadratic equation is the main equation. The interval where a quadratic equation with a positive leading coefficient is in between the two solutions of the quadratic equation. Therefore, let's find the solutions to the quadratic equation. Note that $A = 1$, $B = -3a + 2$, and $C = 2a^2 - 4a$ (A, B, and C are the coefficients of x^2, x^1, and x^0 respectively). Using the quadratic equation (because factoring doesn't work) we get:

- $x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$ | original equation
- $x = \frac{3a - 2 \pm \sqrt{9a^2 - 12a + 4 - 8a^2 + 16a}}{2}$ | plugging in $A = 1$, $B = -3a + 2$, and $C = 2a^2 - 4a$
- $x = \frac{3a - 2 \pm \sqrt{a^2 + 4a + 4}}{2}$ | combining like terms
- $x = \frac{3a - 2 \pm \sqrt{(a+2)^2}}{2}$ | factoring
- $x = \frac{3a - 2 \pm (a+2)}{2}$ | simplifying
- $x = 2a, a - 2$ | simplifying

Thus, the two points where the quadratic equation is 0 is at $2a$ and $a - 2$. We want the distance between these two points to be five. Therefore, either of the following two equations is possible:

$$2a = a - 2 + 5 \text{ or } 2a = a - 2 - 5$$

Solving for a in both of the equations, we get $a = 3$ and $a = -7$. If $a = 3$, then the interval will be inbetween $2a = 6$ and $a - 2 = 1$. However, this is outside the $x < 3$ domain so it does not work. If $a = -7$, the interval will be inbetween $2a = -14$ and $a - 2 = -9$. This is inside the domain $x < 3$. Now, there is one more possibility: the interval ends at $x = 3$. If when one of the solutions is $x = 3$ the interval is > 5, that would mean that there is an additional solution:

$$2a = 3,\ a = \frac{3}{2} \text{ and } a - 2 = 3,\ a = 5$$

At $a = \frac{3}{2}$ the other 0 that isn't 3 is $a - 2 = -\frac{1}{2}$. The interval from $-\frac{1}{2}$ to 3 is not > 5, so there are no more solutions. (There is no need to test $a = 5$ since we already found that the right most 0 at $x = 3$ occurs when $a = \frac{3}{2}$, so for the other a value the 0 $x = 3$ will be the left most 0, which is not possible). Thus, our only answer is $a = -7$.

Competition Answer Key:

Central Jersey Mathematics League Contest [Solutions]

January 14, 2026

1. Let N be a positive integer with exactly 4 distinct prime factors, each of which is raised to the first power. How many positive integers divide N without remainder?

Solution: Let the prime factors be $a, b, c,$ and d. Then N is divisible by 1, N, the prime factors, the product of any two prime factors (ab, ac, ad, bc, bd, cd), and the product of any three prime factors (abc, abd, acd, bcd). The total number of factors is $1 + 4 + 6 + 4 + 1 = 16$. **Answer: 16.**

2. Alice and Bob play the following game. Laying on the table are 26 stones. The players alternately remove 1, 2, 3, 4, or 5 stones from the table, with Alice going first. Whoever takes the last stone wins. There is one value of k such that Alice can guarantee a win by taking k stones on her first turn and then playing perfectly. What is the value of k?

Solution: If Alice leaves a multiple of 6 stones on the table after her turn, then by taking $6 - b$ stones if Bob then takes b stones, she will again leave a multiple of 6 stones on the table. On her last turn, Alice will leave no stones and therefore win. To guarantee a win, Alice should take $k = 2$ stones, leaving $24 = 4 \cdot 6$ stones, at her initial turn; if she does anything else, Bob can win by adopting this same strategy. **Answer: 2.**

3. Real numbers a and b are chosen with $1 < a < b$ such that there are no triangles with positive area whose sides have lengths 1, a, and b or $\frac{1}{a}$, $\frac{1}{b}$, and 1. What is the smallest possible value of b?

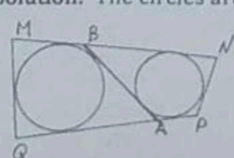
Solution: There is a triangle with side lengths 1, a, and b if and only *if* $a > b - 1$. There is a triangle with side lengths $\frac{1}{a}$, $\frac{1}{b}$, and 1 if and only if $\frac{1}{a} > 1 - \frac{1}{b}$, that is $a < \frac{b}{b-1}$. Therefore, there are no such triangles if and only if $b - 1 \le a \le \frac{b}{b-1}$. The smallest possible value of b satisfies $b - 1 = \frac{b}{b-1}$ or $b^2 - 3b + 1 = 0$. The solution with $b > 1$ is $\frac{3+\sqrt{5}}{2}$. The corresponding value of a is $\frac{1+\sqrt{5}}{2}$. **Answer:** $\frac{3+\sqrt{5}}{2}$.

4. Let $\lfloor x \rfloor$ be the greatest integer less that or equal to x. For $0 < x \le 100$, the values of x satisfying $\lfloor \log_{10}(x+2) \rfloor = \lfloor \log_{10} x \rfloor + 2$ can be written as $a \le x < b$. Find a and b. Present as an ordered pair (a, b).

Solution: Let $\log_{10}(x+2) = A$. Note that for $0 < x < 8$, $0 < A < 1$, so $\lfloor A \rfloor = 0$. Making the left and the right side of the given equation equal, we have $\lfloor \log_{10} x \rfloor + 2 = 0$ where $-2 \le \log_{10} x < -1$, and that is true for $10^{-2} \le x < 10^{-1}$. Those are the only solutions, since for $8 \le x < 10$, $1 < A < 2$, $\lfloor A \rfloor = 1$. The left side of the given equation is 1 while the right side equals 2. For $10 \le x < 98$, the left side of the given equation equals 1 while the right side equals 3, and for $98 \le x < 100$, the left side equals 2 while the right side equals 3 . For $0 \le x < \frac{1}{100}$, the left side equals 0, while $\lfloor \log_{10} x \rfloor$ takes on values of $-3, -4, \ldots$, so that the right side takes on values of $-1, -2, \ldots$. Thus, the only (a, b) is $(10^{-2}, 10^{-1})$. **Answer:** $(\frac{1}{100}, \frac{1}{10})$.

5. In quadrilateral $MNPQ$, two non-intersecting circles are located in such a way that one circle touches $\overline{MN}$, $\overline{NP}$, and $\overline{PQ}$, and the other circle touches $\overline{MN}$, $\overline{MQ}$, and $\overline{PQ}$. Points B and A lie on $\overline{MN}$ and $\overline{PQ}$ respectively, and $\overline{AB}$ is tangent to both circles. Compute MQ in terms of b and p, where $b = NP$, and the perimeter of quadrilateral $BAQM$ is greater than the perimeter of quadrilateral $BAPN$ by $2p$. Express your answer in terms of only b and p.

Solution: The circles are inscribed in quadrilaterals $BAQM$ and $BAPN$ (Fig. 1). The tangent segments are



congruent: $MQ + AB = \frac{1}{2}P_1$, and $AB + NP = \frac{1}{2}P_2$ where P_1 and P_2 are perimeters of quadrilaterals $BAQM$ and $BAPN$. Thus $MQ - NP = \frac{1}{2}(P_1 - P_2) = p$. From here, $MQ = NP + p = b + p$. **Answer: $b + p$.**

6. The equation $x \ln 27 \log_{13} e = 27\log_{13} y$ has integer solutions (x, y). Find the integer solution with the smallest value of y that exceeds 70 and present your answer as $x + y$.

Solution: The original equation is equivalent to equation $\frac{x \ln 27}{\ln 13} = \frac{27 \ln y}{\ln 13} \Leftrightarrow x \ln 27 = 27 \ln y \Leftrightarrow$ $\ln 27^x = \ln y^{27} \Leftrightarrow 27^x = y^{27} \Leftrightarrow 3^x = y^9$, where $y \geq 1$. Since 3 is a prime number, then either $y = 1$ (then $x = 0$), or y has 3 as the only prime factor, That means that $y = 3^n$ and $x = 9n$, where $n \geq 0$, Since $y > 70$, then $n \geq 4$ and the desired solution is $(x, y) = (36, 81)$, then $x + y = 36 + 81 = 117$. **Answer: 117.**

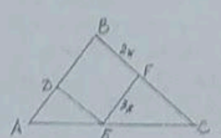
7. Aaron took 30 numbers, rounded each of them up to the nearest integer, added the results, and got 2026. When he took the original numbers, rounded each of them down to the nearest integer and added the results, he got 2010. How many of his original numbers were integers?

Solution: If any of Aaron's 30 original numbers (let it be x) was an integer, then its value did not change when rounding up or down to the nearest integer. Therefore, $\lceil x \rceil$ contributed to the first term as much as $\lfloor x \rfloor$ contributed to the second term. If any of Aaron's 30 original numbers (let it be x) was not an integer, then its value increased (by less than one) when rounded up to the nearest integer and decreased (by less than one) when rounded down to the nearest integer. The results of these two rounding operations are consecutive integers. (Formally, $\lfloor x \rfloor < x < \lceil x \rceil$, and $\lceil x \rceil - \lfloor x \rfloor = 1$ for any non-integer x). Therefore, $\lceil x \rceil$ contributed to the first sum 1 more than $\lfloor x \rfloor$ contributed to the second sum. Thus, the value of the first sum (2026) is greater than the value of the second sum (2010) by the total number of non-integers. Therefore, there were $2026 - 2010 = 16$ non-integers and $30 - 16 = 14$ integers among these 30 numbers. **Answer: 14.**

8. Solve $1 + 2^{\tan x} = 3 \cdot 4^{\left(\frac{\sin(\frac{\pi}{4} - x)}{\sqrt{2}\cos x}\right)}$ where $0 < x \leq 2\pi$.

Solution: Note that $\frac{\sin(\frac{\pi}{4} - x)}{\sqrt{2}\cos x} = \frac{\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x}{\sqrt{2}\cos x} = \frac{1 - \tan x}{2}$. Let $2^{\tan x} = y$. Then we get the following equation: $1 + y = 3 \cdot \frac{2}{y}$. Solving for y: $y^2 + y - 6 = 0$, $y = 2, -3$. Since $y > 0$, $y = 2$, then we get $\tan x = 1$, and after solving for x: $x = \frac{\pi}{4}, \frac{5}{4}\pi$. **Answer:** $\frac{\pi}{4}, \frac{5}{4}\pi$.

9. In isosceles triangle ABC, point E is located on the base $\overline{AC}$, and points D and F are located on the lateral sides $\overline{AB}$ and $\overline{BC}$, respectively. $\overline{DE}$ is parallel to $\overline{BC}$ and $\overline{EF}$ is parallel to $\overline{AB}$. What part of the area of triangle ABC is the area of triangle DEF if $BF : EF = 2 : 3$?



Solution: Quadrilateral $BDEF$ is a parallelogram. Triangles ADE and EFC are similar to triangle ABC (Fig. 2): $\frac{DE}{BC} = \frac{2x}{2x+3x} = \frac{2}{5}$ and $\frac{FC}{BC} = \frac{3x}{2x+3x} = \frac{3}{5}$, respectively. Then $\frac{A_{ADE}}{A_{ABC}} = \frac{4}{25}$ and $\frac{A_{EFC}}{A_{ABC}} = \frac{9}{25}$. It follows $A_{DEF} = \frac{1}{2}A_{BDEF} = \frac{1}{2}(A_{ABC} - A_{ADE} - A_{EFC}) = \frac{1}{2}\left(1 - \frac{4}{25} - \frac{9}{25}\right)A_{ABC} = \frac{6}{25}A_{ABC}$. **Answer:** $\frac{6}{25}$.

10. Find real a such that the solution of the following system is an interval of length 5:

$$\begin{cases} x^2 - x(3a - 2) + 2a^2 - 4a < 0 \\ x < 3 \end{cases}$$

Solution: The real solutions of the quadratic equation on the left side are $x_1 = 2a$ and $x_2 = a - 2$. The solution of the quadratic inequality is one of the intervals $(2a, a - 2)$ or $(a - 2, 2a)$. The given system has a solution if number 3 is located on the right side from the left end of the interval. We also need to check whether 3 is on the right side of the right end of the interval. Case 1: If $2a < a - 2$, i.e. $a < -2$, then $a - 2 < -4 < 3$, so the solution to the system is the interval $(2a, a - 2)$. In this case, $a - 2 - 2a = 5$, and it follows that $a = -7$. Case 2: If $a - 2 < 2a$, i.e. $a > -2$, then a could be found from $3 - (a - 2) = 5$ or from $2a - (a - 2) = 5$. Solving these equations, $a = 0$, $a = 3$. Let's test the obtained values $a = 0$, and $a = 3$ by substituting them into the original system. The quadratic inequality, when $a = 0$, produces the interval $(-2,0)$; when $a = 3$, it produces the interval $(1, 6)$. The first obtained interval does not even have length 5, thus $a = 0$ must be rejected. The second interval $(1, 6)$ in combination with $x < 3$ produces the overlap interval $(1, 3)$ which does not have length 5 either, therefore, $a = 3$ must be rejected. The only value of a that works is $a = -7$.
Answer: −7.