Please attempt the problems on your own before consulting the answer key!

1. **In the coordinate plane, $A = (1, -2)$ and $B = (6, 10)$. Find the point P on the segment AB which is twice as far from A as it is from B. Express your answer as an ordered pair.**

Answer: $(\frac{13}{3}, 6)$

Solution: Because point P is twice as far from A as it is from B, we can say that that the lengths of segments AP and PB are in the ratio $2 : 1$. Therefore, the distance from point A to point P is twice the distance from point P to point B (segment AP is twice the length of segment PB). From this, we can set up an equation to solve for segment PB (we could also solve for AP first):

- $AB = AP + PB$ | original equation
- $AB = 2PB + PB$ | substituting AP with $2PB$ because we know AP is twice PB
- $AB = 3PB$ | combing like terms
- $\frac{1}{3}AB = PB$ | solving for segment PB

Thus, we know that segment PB is $\frac{1}{3}$ the length of segment AB. Since segment AB is 5 units in the x-direction and 12 units in the y-direction, segment PB will be $\frac{1}{3} * 5 = \frac{5}{3}$ units in the x direction and $\frac{1}{3} * 12 = 4$ units in the y direction. Point B is right and up of point A, so a point between points B and A will be left and down from point B (and right and up from point A). Therefore, we subtract the x and y directions we found earlier from point B to find point P:

- x direction: $6 - \frac{5}{3} = \frac{13}{3}$
- y direction: $10 - 4 = 6$

So, the point P will have the coordinates (as an ordered pair x,y) of $(\frac{13}{3}, 6)$. Our answer is $(\frac{13}{3}, 6)$.

2. **The president of Dollarstan is weighing two income tax plans. In the first plan, all residents would pay tax 10% of their yearly income. According to the second plan, the residents would pay tax 16% of any yearly income over $\$150,000$. The President noticed that his own tax under either plan is the same. What is the yearly income of the President of Dollarstan?**

Answer: $\$400,000$

Solution: First off, I'd like to start by saying that anyone who answered $\$0$ (I'm looking at you Avaneesh) is technically not wrong, but you would not get the point for this question unfortunately. The question, possibly unfairly, implies that the President's income is $> \$0$ (he has a salary), therefore invalidating $\$0$ as a possible answer.

Now, let's move to solve this problem; there are two tax plans. To start, let's create an equation for the first tax plan: in it, residents pay 10% of their yearly salary as tax. Letting x be the yearly salary, the tax, t, that any person would have to pay in this tax plan is $t = \frac{10}{100} * x = \frac{x}{10}$. The second tax plan states that any income over $\$150,000$ would be subject to a 16% tax. Therefore, using the same variables from before, we can say $t = \frac{16}{100} * (x - 150000) = \frac{4}{25} * (x - 150000)$. The tax is changed to 16% and $150,000$ is subtracted from the yearly salary, x, because we only want to count money over $\$150,000$. Because our equation implies that incomes $< \$150,000$ garner a negative tax, we must be vigilant of extraneous solutions (negative or zero salaries). Nonetheless, moving on, we know that the President's salary under each tax plan is the same so we can set the tax plans equal to each other and solve (without a calculator) for the yearly salary, x:

- $\frac{x}{10} = \frac{4}{25} * (x - 150000)$ | original equation

- $5x = 8 * (x - 150000)$ | multiplying both sides by 50 to get rid of the fractions
- $5x = 8x - 8 * 150000$ | expanding the right side
  - Note that although you can simplify $8 * 150000$, it is a bit risky, so I decided not to
- $-3x =- 8 * 150000$ | bringing x terms to one side
- $x = 8 * 50000$ | dividing both sides by -3
- $x = 400000$ | simplifying the right side

Thus, the President's salary is $\$400,000$. Feel free to check the answer during the competition, but know that you will lose time while doing so. Also note that although the "official" answer is $\$400,000$, I expect that they would've also accepted answers such as "400000," "$\$400000$," and "$400,000$." Nonetheless, our answer is $\$400,000$.

3. **In a given sequence, $a_1 = 2, a_2 = 3,$ and the rule is: $a_{n+2} = a_{n+1}/a_n$ where $n = 1, 2, 3,$ .... Find a_{2025}.**

Answer: $\frac{3}{2}$

Solution: Let's start by examining the rule. It states that for any term in the given sequence, the term two ahead of it (a_{n+2}) is equal to the term one ahead of it (a_{n+1}) divided by itself (a_n). Ok, let's try applying this to try and find some terms. Starting with a_1, we can say that the term two ahead of it ($a_{1+2} = a_3$) is equal to the term one ahead of it ($a_{1+1} = a_2$) divided by itself (a_1). Here it is step-by-step:

- $a_{n+2} = a_{n+1}/a_n$ | original equation
- $a_{1+2} = a_{1+1}/a_1$ | substituting $n = 1$ into the equation
- $a_3 = a_2/a_1$ | simplifying
- $a_3 = 3/2$ | substituting in given values $a_2 = 3$ and $a_1 = 2$

Thus, we figured out that $a_3 = \frac{3}{2}$. From here, we can continue this process for $n = 2$:

- $a_{n+2} = a_{n+1}/a_n$ | original equation
- $a_{2+2} = a_{2+1}/a_2$ | substituting $n = 2$ into the equation
- $a_4 = a_3/a_2$ | simplifying
- $a_4 = \frac{3}{2}/3$ | substituting in given values $a_2 = 3$ and $a_1 = 2$
- $a_4 = \frac{3}{6} = \frac{1}{2}$ | simplifying

Since we are trying to find a_{2025}, an extraordinarily large term number, we can safely assume that there will some sort of pattern or way to find a_{2025} without having to go one-by-one through every single term. Keep this in mind as I lay out the next few terms (during the competition you'd have to perform the same process to find them):

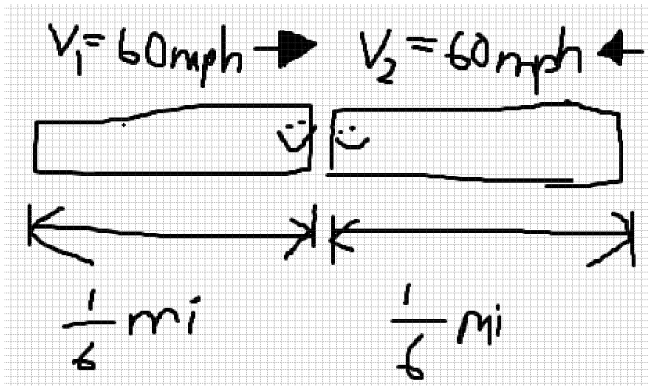
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
2	3	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	2	3	$\frac{3}{2}$

As shown in the table, the sequence begins to repeat at the seventh term. Therefore, the sequence cycles through the same 6 values. Because the sequence cycles through the same 6 terms, we can say that a_{2025} will have the same value as the term $a_{2025\,\%\,6}$. (% represents the modulus operation, please look into it! For a quick explanation, it basically divides a number and finds the remainder.) In this case, $2025 \equiv 3 \pmod 6$. (During the competition, you'd have to perform long division to find the remainder.) Thus, $a_{2025} = a_3 = \frac{3}{2}$. Our answer is $\frac{3}{2}$.

4. **Two freight trains, each $1/6$ miles long and traveling 60 miles per hour, meet and pass each other. How many seconds is it from the moment when the locomotives pass each other to the moment when the rear cars (the cabooses) pass each other?**

Answer: 10 **seconds**

Solution: From fear of retaliation from Mr. Rollis, let's draw a diagram. Make it as crude as possible! In my diagram, I have two rectangles denoting the two trains, each labeled as going 60mph (in opposite directions) and $\frac{1}{6}$ of a mile long:



(Feel free to compliment my artistic abilities)

We are asked to find how much time it takes from when the locomotives (which are the fronts of the trains for those who aren't born in the 1900s) pass each other to when the cabooses (which are the backs of the trains) pass each other. Although a diagram is in no way required, it can be extremely helpful if you don't immediately understand the question. Since the trains are moving at the same speed, their rear ends (haha) will meet at the halfway point between them. The entire distance between the cabooses is $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$ mile, and divided by 2 (because they are both going only halfway) is $\frac{1}{3} \div 2 = \frac{1}{6}$ mile. The question asks for the amount of seconds, so we have to convert the 60 mph into miles per second. Time for some dimensional analysis:

$$\frac{60mi}{1h} * \frac{1h}{60min} * \frac{1min}{60sec} = \frac{1mi}{60sec}$$

So, 60 mph is the same as $\frac{1}{60}$ of a mile per second. Knowing that speed = distance ÷ time, and that therefore time = distance ÷ speed, we can solve for the time because we know both the distance ($\frac{1}{6}$ mile) and the speed ($\frac{1}{60}$ miles per second). Now we just plug and chug:

- time = distance ÷ speed | original equation
- time = $\frac{1mi}{6} \div \frac{1mi}{60sec}$ | plugging in the distance and speed
- time = $\frac{\frac{1mi}{6}}{\frac{1mi}{60sec}}$ | making one fraction to make the next steps easier
- time = $\frac{10mi}{\frac{1mi}{1sec}}$ | multiplied top and bottom of the fraction by 60
- time = $\frac{10}{\frac{1}{1sec}}$ | divided top and bottom by mi (or $1mi$)
- time = $\frac{10sec}{\frac{1}{1}}$ | multiplied top and bottom by sec (or $1sec$)

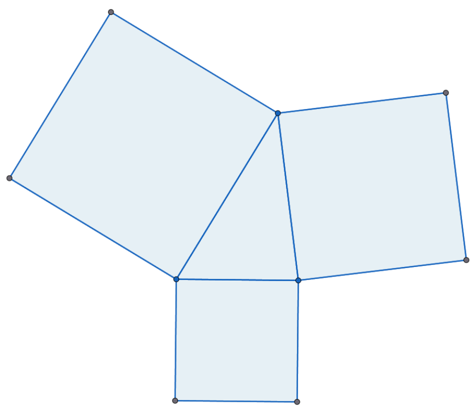
- time $= 10sec$ | simplify

Thus, the time between when the locomotives pass each other and when their cabooses pass each other is 10 seconds. Our answer is 10 seconds.

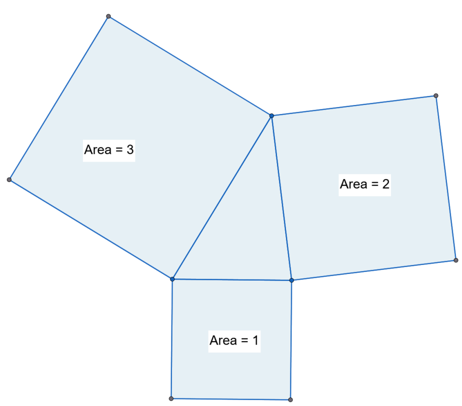
5. **A square is drawn on each side of a triangle (each side of the triangle is a side of one of the squares). None of the four shapes overlap. The ratio of the areas of the squares is $1:2:3$. What is the degree measure of the largest angle of the triangle?**

Answer: $90°$

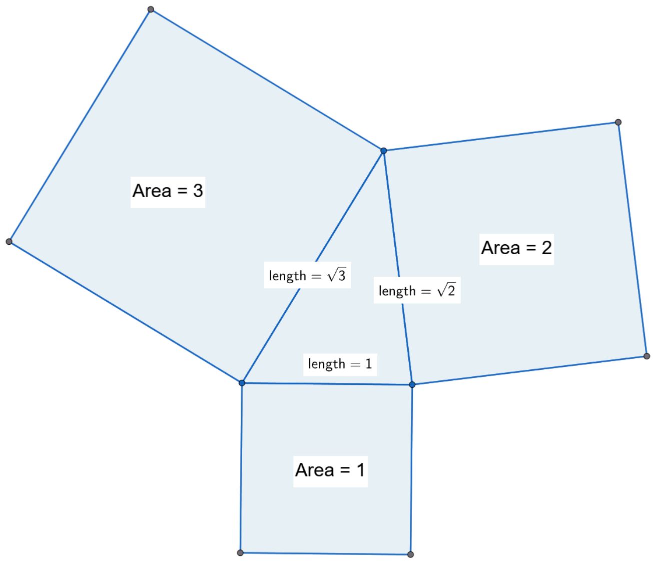
Solution: Let's make a diagram. There is no need for it to be to scale (but making it reasonable makes it easier). We have a triangle in which each side of the triangle corresponds to a side of a square:



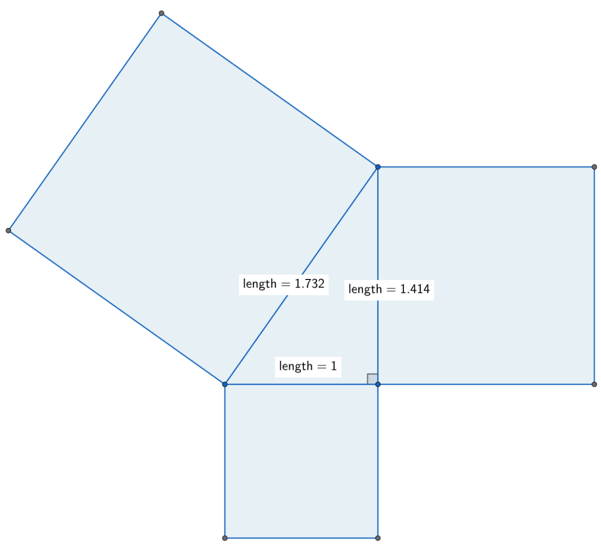
Our diagram satisfies the rule that none of the four shapes overlap (although the triangle shares sides with the squares, the sides are not considered overlapping). Next, the problem states that the ratios of the squares are $1:2:3$. Since it doesn't give us specific areas but rather just the ratio between the areas, let's try values that satisfy this ratio and see what we can find. Because it's easy, let's use areas of $1, 2,$ and 3:



Knowing that the length of a side of a square in the square root of its area, the sides of the squares (and triangle) are $1, \sqrt{2}$, and $\sqrt{3}$:



From here, notice that the lengths of the sides of the triangle form a (non-integer) pythagorean triple. This means that the side opposite the hypotenuse, the longest side, will be $90°$. Keep in mind that the angle opposite the side with length $\sqrt{3}$ may not seem to be $90°$ because the drawing was not to scale. However, because we estimated each side reasonably, we can tell that the angle is ~$80°$, so it is not surprising that the angle is actually $90°$. Here is the to scale drawing for reference:



Note that $\sqrt{3} \approx 1.732$ and $\sqrt{2} \approx 1.414$. Moving on, because we know that one angle of the triangle is $90°$, we also know that no other angle in the triangle can be $\geq 90°$ because the sum of all the angles in a triangle is $180°$, and if one angle is $90°$, then the other two must sum to $90°$. Since an angle of a triangle cannot be $0°$ (because then it'd just be a line), both of the other two angles must be $< 90°$. Thus, $90°$ will be the degree measure of the largest angle in the triangle. Our answer is $90°$.

6. **It is known that in some math class, Al, Sam, and Mike are the same height. The average height of all students in the class except for Al is 63.1 inches. The average height of all students in the class except for both Al and Sam is 62 inches. The average height of all**

students in the class except for all three of them (Al, Sam, and Mike) is 60.625 inches. How many students are in this class?

Answer: 11

Solution: The number of students in class must be an integer, so the average of all the students' heights will be the total of all the students' heights divided by an integer number of kids. Keeping this in mind, let's look at the averages after each student in excluded: 63.1, 62, and 60.625. 63.1 has a fractional component of $\frac{1}{10}$, 62 does not have a fractional component, and 60.625 has a fractional component of $\frac{5}{8}$ (this might be a little harder to notice, but with time it will be easy to spot). Since the number of kids which divided the total height must be an integer, it is likely that the the average height of 63.1 had 10 students because the fractional component has a denominator of 10 and that the average height of 60.625 had 8 students because the fractional component has a denominator of 8. This would also make sense because the average height of 60.625 that we think has 8 students came after two students were removed and we think the average height of 63.1 has 10 students. However, before jumping to conclusions, let's check to make sure. Given the average height of every student in the class, we can multiply it by the number of students in the class to get the total height of all the students:

- $63.1 * 10 = 631$
- $62 * 9 = 558$
- $60.625 * 8 = 485$

Al, Sam, and Mike are the same height, so removing them each one by one would drop the total heights of all the students by a constant amount. So, if there is a constant difference between each of the total heights, then we are correct with the number of students.

$$\begin{aligned} &\bullet\ 63.1 * 10 = 631 \\ &\qquad\qquad\qquad\quad \big\} -73 \\ &\bullet\ 62 * 9 = 558 \\ &\qquad\qquad\qquad\quad \big\} -73 \\ &\bullet\ 60.625 * 8 = 485 \end{aligned}$$

Thus, we are correct with the number of students in the class. The question asks for the number of students in the class (so without anyone removed). Because we know that the first average (63.1) is of everyone in the class except for Al, and that there were 10 students in that average, we just add 1 (for Al because he was excluded) to get the total number of students in the class: $10 + 1 = 11$. Our answer is 11.

7. **Find the positive square root of the sum of the cubes of the first 15 natural numbers.**

Answer: 120

Solution: Yes, you can brute force it. No, I will not endorse it. Please do not sum the cubes of the first fifteen natural numbers $(1, 2, 3, ..., 15)$ and take the square root of it. It's not worth it.

Swiftly moving on, let's try looking for a pattern. Looking for a pattern is often the best course of action for these kinds of problems. Let's calculate the positive square root of the sum of the cubes for the first few natural numbers:

natural number (k)	cubed (k^3)	sum of it and previous cubes ($\sum_{n=1}^{k} n^3$)	positive (aka principal) square root ($\sqrt{\sum_{n=1}^{k} n^3}$)
1	$1^3 = 1$	$1 = 1$	$\sqrt{1} = 1$

2	$2^3 = 8$	$8 + 1 = 9$	$\sqrt{9} = 3$
3	$3^3 = 27$	$27 + 8 + 1 = 36$	$\sqrt{36} = 6$
4	$4^3 = 64$	$64 + 27 + 8 + 1 = 100$	$\sqrt{100} = 10$

Don't worry if you don't understand the Σ notation. It's honestly there just because I think it looks cool, but I nonetheless encourage you to learn it! Anyways, let's look to see if we can find a pattern. We have the integers $1, 3, 6, 10$. Perhaps you already see it, but in case not, let's find the pattern manually. The difference between the first and second terms is 2, then the second and third is 3, then the third and fourth is 4. We can therefore guess that the difference between the fourth and the fifth terms is 5, and that the fifth term is therefore the fourth term plus five: $10 + 5 = 15$. But let's just make sure:

5	$5^3 = 125$	$125 + ... + 1 = 225$	$\sqrt{225} = 15$

We are indeed correct. Now, you could sum:

$$15 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 = 120,$$

But it is a little bit time consuming (but still not as bad as the earlier brute force method). Instead, it's a lot faster to do one of two things:

- Use the sum of an arithmetic sequence formula: $n(\frac{a_1+a_n}{2})$
  - $15(\frac{1+15}{2}) = 120$
  - The number of terms, $n = 15$, times the sum of the first term, $a_1 = 1$, and the last term, $a_n = 15$, divided by 2
- Use the sum of natural numbers formula: $n(\frac{1+n}{2})$
  - $15(\frac{1+15}{2}) = 120$ (yes, it's the same as the previous one)
  - Notice that the sum of natural numbers formula is the same as the sum of an arithmetic sequence formula, but a_1 is assumed to be 1, and a_n is assumed to be equal to n (because the nth natural number is n)
  - To have used this formula, you must have noticed that $1, 3, 6, 10, ..., a_n$ is just the sum of natural numbers from 1 to n.

During the competition I actually used neither of these formulas because I was in a rush and missed them. Luckily, they weren't necessary for this competition. However, imagine if the question said instead the first 50 natural numbers. Summing it up manually would have been a lot more difficult and time consuming. Nonetheless, we got our answer of 120.

8. **Find the real numbers a and b such that any pair (x, y) satisfying the equation $x/y = a$, satisfies the equation $(x + y)^2 = b$. Present your answer as an ordered pair (a, b).**

Answer: $(-1, 0)$

Solution: Ok, we need to find two constants, a and b such that if $x/y = a$, then $(x + y)^2 = b$. Let's try some test numbers to try and find what a and b must be. I will let $x = a$, and $y = 1$. Because the first equation, $x/y = a$ is satisfied (because $a/1 = a$), the second equation must also be satisfied. Therefore, $(a + 1)^2 = b$. Seeing as we have two variables, let's try another test case to see if we can create a system of equations so that we can solve for a and b. I will now let $x = 2a$ and $y = 2$. This satisfies $x/y = a$ (because $2a/2 = a$), so it must also satisfy $(x + y)^2 = b$. Therefore, $(2a + 2)^2 = b$ must also hold true. Now that we have two equations that have b isolated, let's set substitute in b in the equations to solve for a:

- $(a + 1)^2 = b$ and $(2a + 2)^2 = b$ | original equations
- $(a + 1)^2 = (2a + 2)^2$ | substitution $(2a + 2)^2$ for b in the first equation
- $a + 1 = 2a + 2$ | taking the square root of both sides
- $- a = 1$ | moving a to the left side
- $a =- 1$ | dividing both sides by $- 1$ to find a

So now that we know a must be $- 1$, we can solve for b by plugging a back into either equation:

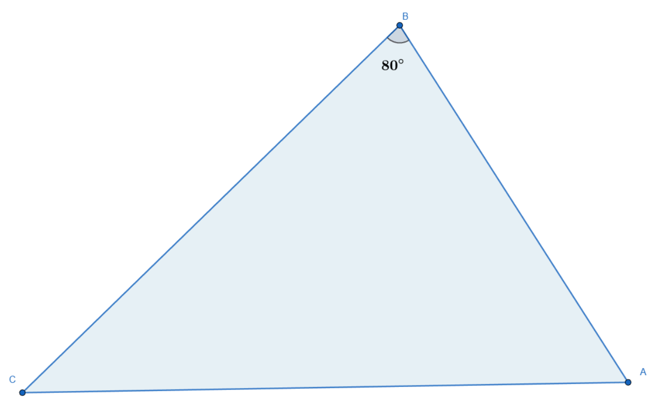
- $(a + 1)^2 = b$ | original equation
- $(- 1 + 1)^2 = b$ | substituting in $a =- 1$
- $0 = b$ | evaluating the left side

Thus, we've found the values for both a and b. I'd recommend testing the answer in case it is an extraneous solution. In this case it isn't. The question asks us to put our answer in the form (a, b), so our answer is $(- 1, 0)$.

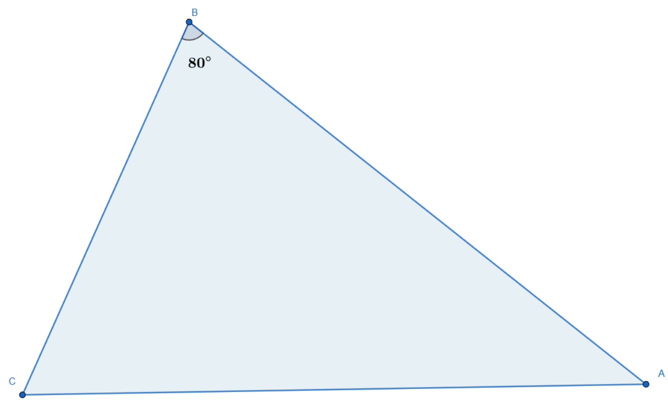
9. **In triangle ABC, $m\angle B = 80°$. Point D is located on segment BC, and segment AB ≅ segment AD ≅ segment CD. Point F is located on segment AB, and segment AF ≅ segment BD. Point E is located on segment AC, and segment AB ≅ segment AE. Find $m\angle AEF$.**

Answer: $20°$

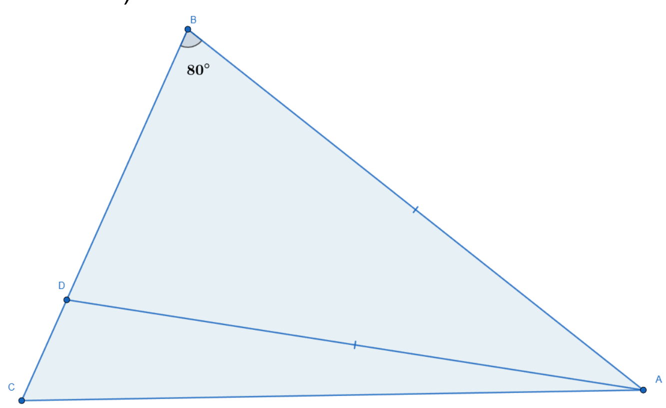
Solution: This seems like a complicated geometry problem, so let's start by creating a diagram. The question tells us that in triangle ABC, $m\angle B = 80°$, so let's try to make a triangle ABC that has an angle B that is about $80°$ (for now the other angles can be random):



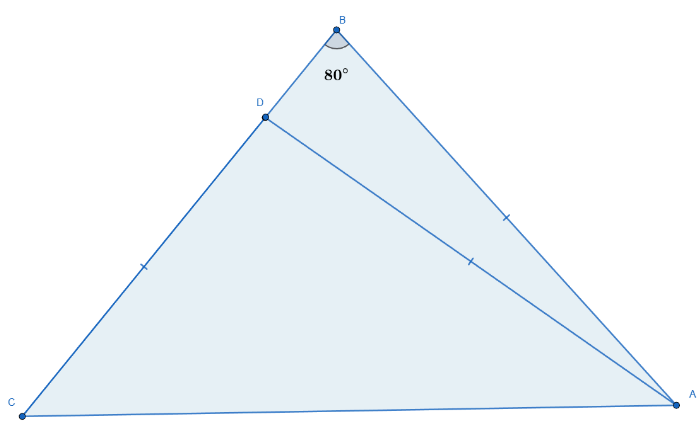
Angle B is not actually $80°$ in my diagram, but it's a pretty good estimate. In regards to the other measures and side lengths of the triangle, we can always change them if we need as we get more parameters from the problem. Moving on, the question tells us that there is some point D on segment BC such that segments AB, AD, and CD are congruent. Because I see that segments AB and AD will need to be congruent, I've moved point B to the left so that the congruence statements will make more sense visually on the diagram. Keep in mind that you will probably have to do this multiple times while solving geometry problems like this one that has many parameters. Here is the updated diagram:



Now, we know that point D must be somewhere on segment BC such that segments AB and AD are congruent. Let's plot point D on segment BC and mark segments AB and AD congruent (with a dash/mark on both lines):

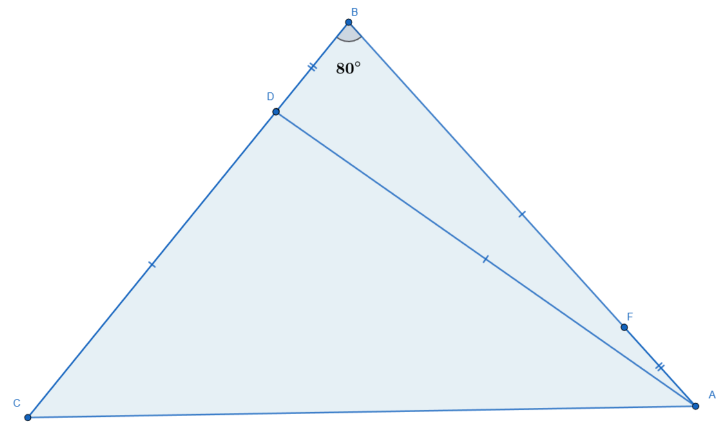


It may not look the most accurate, but it's passable. Next, the question states that segment CD is also congruent to both segments AB and AD, so I'll move point D up and point B a little to the right to make the diagram look more reasonable. I will also mark segment CD as congruent using the same single dash (because all three segments are congruent):

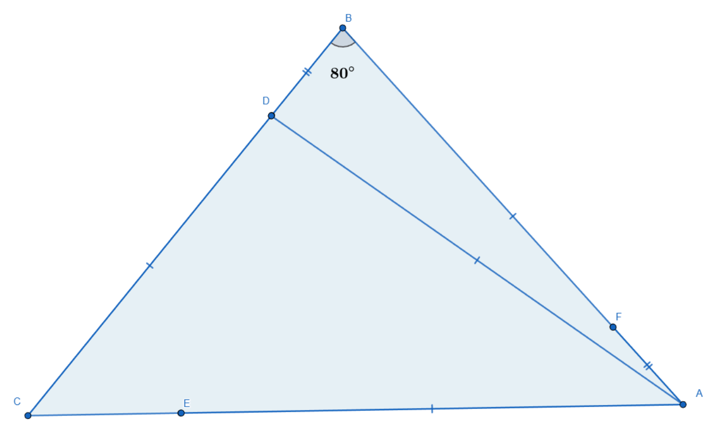


Keep in mind that the dash on the left side marks congruence for segment CD, not segment CB. Anyways, next there is a point F on segment AB such that segments AF and BD are congruent. I'll create a point F

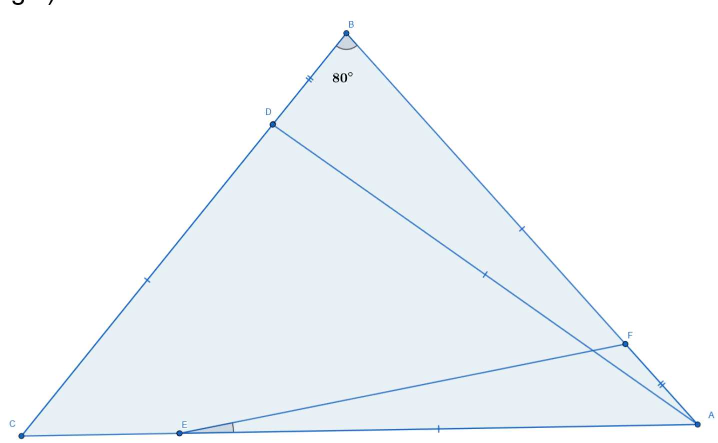
on my diagram and mark segments AF and BD as congruent (with a double dash this time to differentiate their congruence from the other three segments' congruences):



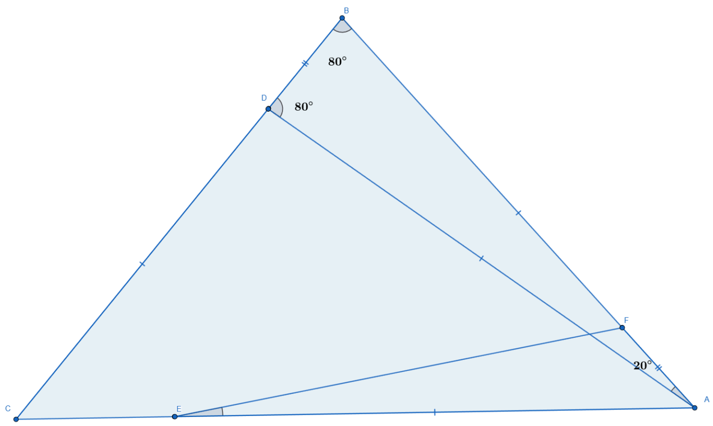
Finally, the problem states that there is a point E on segment AC such that segments AB and AE are congruent. Since segment AB is already congruent to segments CD and AD, that means that segment AE is also congruent to them. As such, we can give segment AE a single dash as well:



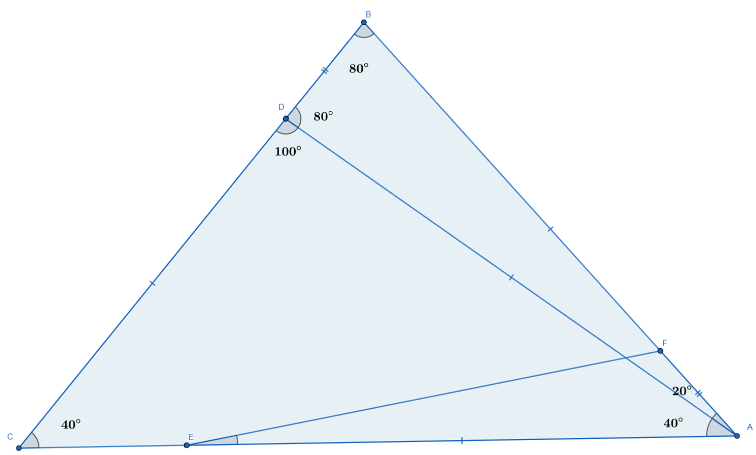
We've finally satisfied all the parameters of the problem. Now, let's mark what we're solving for, $m\angle AEF$, so that we know what we're trying to find (I've also created segment EF to make it easier to mark the angle):



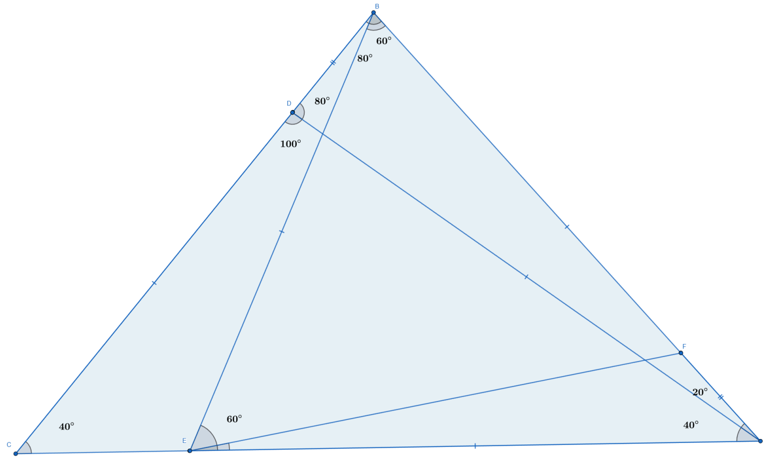
To find the $m\angle AEF$, let's try solving for some other angles so we can eventually find the $m\angle AEF$. First off, notice that triangle BAD is an isosceles triangle because two of its sides, segments BA and DA, are congruent. This means that $m\angle BDA = m\angle DBA = 80°$. In addition to that, since all the angles in a triangle must add up to $180°$, $m\angle BAD = 180° - m\angle BDA - m\angle DBA = 180° - 80° - 80° = 20°$. Let's mark this on our diagram:



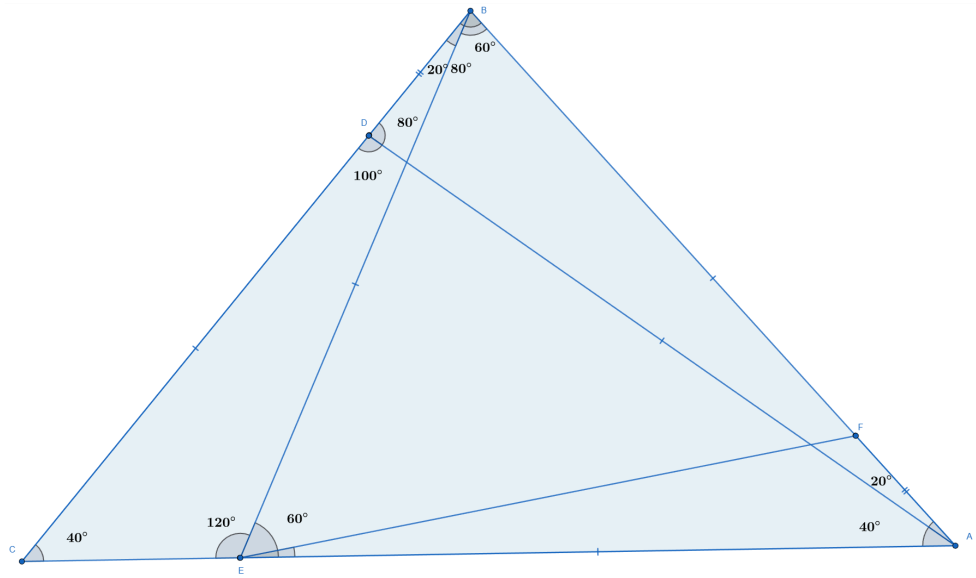
Because $\angle CDA$ and $\angle BDA$ form a linear pair, their angles sum up to $180°$, so $m\angle CDA = 180° - m\angle BDA = 180° - 80° = 100°$. Furthermore, because triangle CDA is an isosceles triangle (because segments DA and DC are congruent), $m\angle ACD = m\angle CAD$. Since we just found that $m\angle CDA = 100°$, the measures of the two other angles in triangle CDA must be $(180° - 100°)/2 = 40°$. Let's mark the measures of all the angles in triangle CDA that we just found on our diagram:



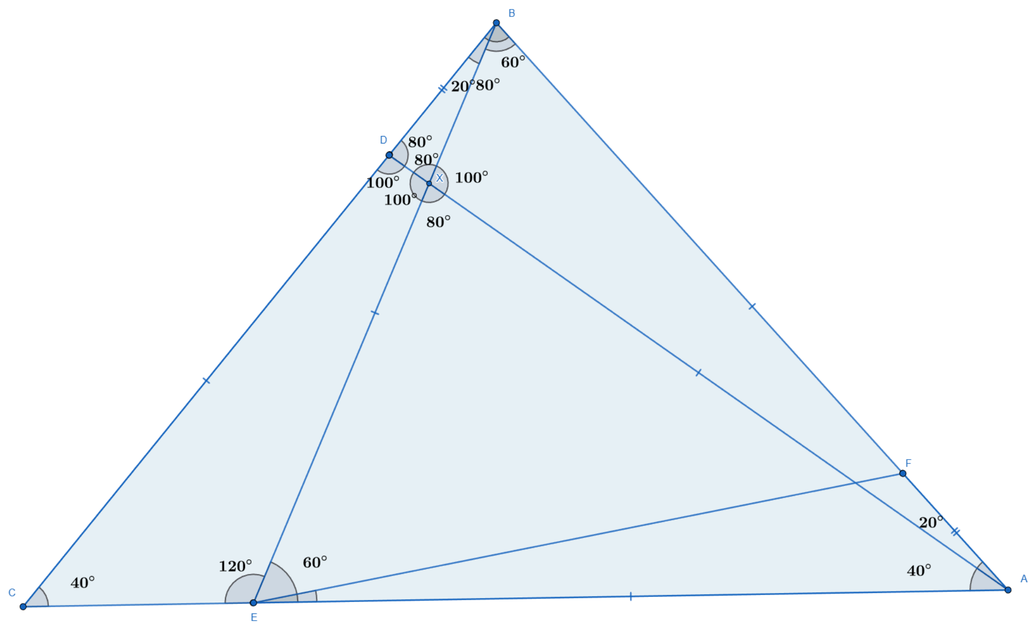
Now comes the hardest part: notice that the $m\angle CAB = 60°$ and that two sides that create the angle, segments AB and AE, are congruent. Because of this, we can construct a segment BE (which is congruent the two segments) to create an equilateral triangle EBA:



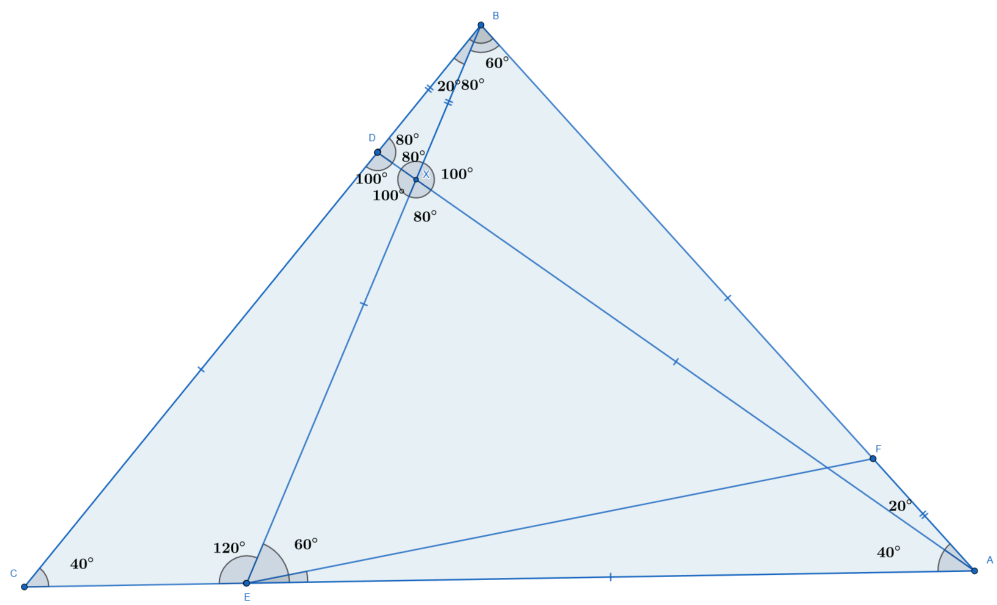
Now, we can label some more angles. $m\angle CBE = m\angle CBA - m\angle EBA = 80° - 60° = 20°$. Additionally, $m\angle CEB = 180° - m\angle AEB = 180° - 60° = 120°$. Let's label these angles on the diagram:



Now we can solve for another four angles. Let's label the intersection of segments BA and AD as point X. Because all the angles in a triangle (in this case, triangle BDX) must sum to $180°$, $m\angle BXD = 180° - m\angle XBD - m\angle XDB = 180° - 20° - 80° = 80°$. Because $\angle BXD$ and $\angle EXA$ are a vertical pair, the measure of their angles are equal so $m\angle EXA = m\angle BXD = 80°$. Then because they are linear pairs with the aforementioned angles, $m\angle DXE = m\angle BXA = 180° - m\angle EXA = 180° - 80° = 100°$. Let's label what we've just found:



Notice that in triangle BDX, $m\angle BDX \cong \angle BXD$, meaning that it is an isosceles triangle and that segments BD and BX are congruent. Let's mark them congruent:



Now it's time for the final step. Notice that $\Delta FAE \cong \Delta XBA$ by SAS congruence (segment $FA \cong$ segment XB, $\angle FAE \cong \angle XBA$, and segment $AE \cong$ segment BA). Because $\angle E$ and $\angle A$ are corresponding angles, $m\angle FEA = m\angle XAB = 20°$. $\angle AEF$ and $\angle FEA$ are the same angle, so $m\angle AEF = m\angle FEA = 20°$. And with that, we've finally obtained our answer of $20°$. (I understand this may be quite confusing. Please ask us if you are confused, or if you didn't understand any of it, tell us and we can help you)

10. **Find all possible values of a such that $x^2 + 2x + a \leq 0$ and $x^2 - 4x - 6a \leq 0$ has one real solution.**

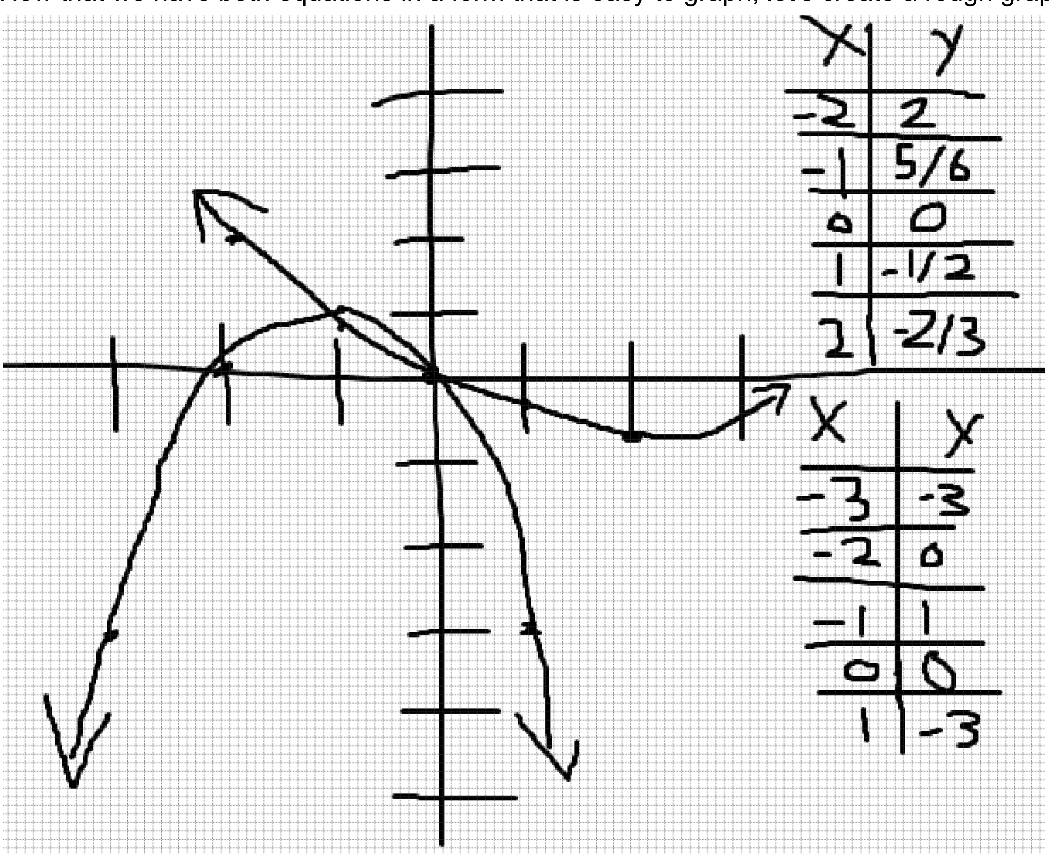
Answer: $1, 0$

Solution: We want to find all values of a such that the two equations have only one common solution. Frankly, I am not sure how to do this algebraically, but the answer key provides a simple way to do it graphically so I will explain it that way. First off, let's solve for a in both equations:

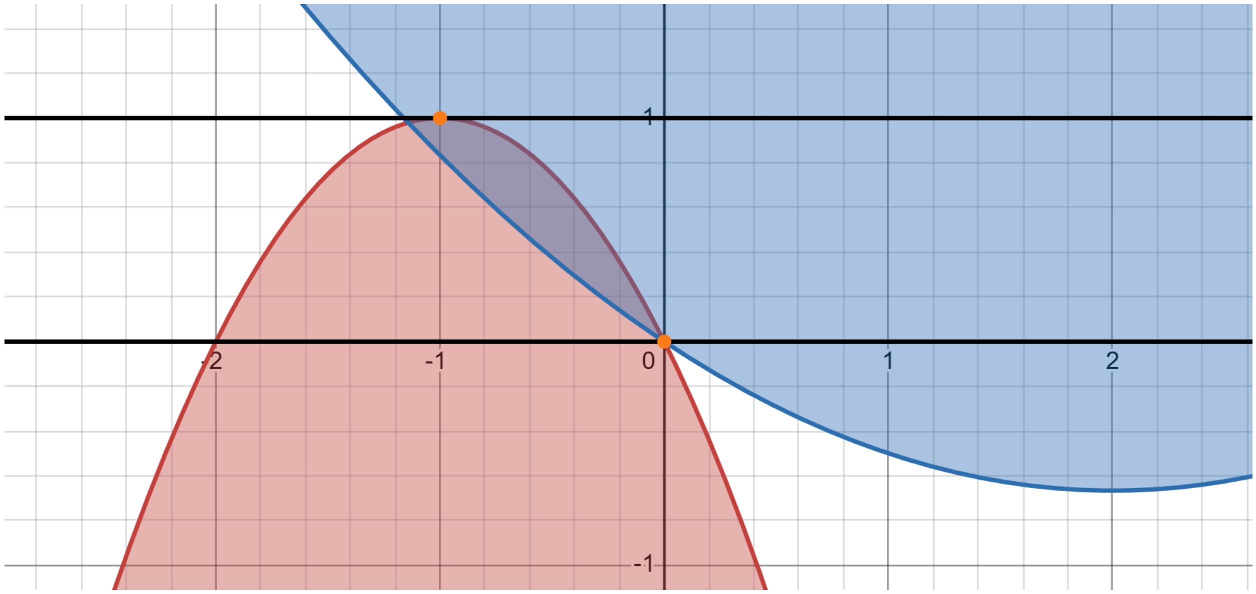
- $x^2 + 2x + a \leq 0$ | original equation
- $a \leq -x^2 - 2x$ | isolating a
- $a \leq -(x^2 + 2x)$ | factoring out -1 to make turning the right side to vertex form easier
- $a \leq -(x^2 + 2x + 1) + 1$ | step to [complete the square]
- $a \leq -(x + 1)^2 + 1$ | factoring the parenthesis (it is now in vertex form)

- $x^2 - 4x - 6a \leq 0$ | original equation
- $-6a \leq -x^2 + 4x$ | bring a to one side
- $a \geq \frac{1}{6}x^2 - \frac{2}{3}x$ | dividing both sides by -6 (remember to flip the sign)
- $a \geq \frac{1}{6}(x^2 - 4x)$ | factoring out $\frac{1}{6}$ to make completing the square easier
- $a \geq \frac{1}{6}(x^2 - 4x + 4) - \frac{2}{3}$ | step to complete the square
- $a \geq \frac{1}{6}(x - 2)^2 - \frac{2}{3}$ | factoring the parenthesis (it is now in vertex form)

Now that we have both equations in a form that is easy to graph, let's create a rough graph of them:



Now let's interpret the graph: everything above the concave up quadratic and below the concave down quadratic is a solution, and we want only one common solution between them. So, any horizontal line that we draw through the graph should have only one common point of intersection between the two functions. The only two places in the common region between the two functions which would create only one point of intersection between a line and the solution is at $y = 1$ and $y = 0$. Thus, the only two values of a that make it so that the system of equations have only one solution are $a = 1, a = 0$. Our answer is $1, 0$. Here is a more clear graph to make seeing this answer easier, but know that you will have to come to the answer with a graph like the one pictured above during the actual competition:



Competition Answer Key:

Central Jersey Mathematics League Contest
[Solutions]
October 15, 2025

1. In the coordinate plane, $A = (1, -2)$ and $B = (6, 10)$. Find the point P on the segment $\overline{AB}$ which is twice as far from A as it is from B. Express your answer as an ordered pair.

Solution: $A = (1, -2)$, and let $P = (x, y)$. Let $x = 1 + \Delta x$, $y = -2 + \Delta y$. Now, $\Delta x = \frac{2}{3}(6-1) = \frac{10}{3}$, and $\Delta y = \frac{2}{3}(10+2) = 8$. Therefore, $x = \frac{13}{3}$, and $y = 6$. $P = \left(\frac{13}{3}, 6\right)$. **Answer:** $\left(\frac{13}{3}, 6\right)$.

2. The President of Dollarstan is weighing two income tax plans. In the first plan, all residents would pay tax 10% of their yearly income. According to the second plan, the residents would pay tax 16% of any yearly income over $150,000. The President noticed that his own tax under either plan is the same. What is the yearly income of the President of Dollarstan?

Solution. Let $x > 0$ be the yearly income of the President of Dollarstan. Under the first plan, the President would pay tax equal to 10% of his yearly income, i.e. $0.1x > 0$. If $x \leq 150{,}000$, then under the second plan the President would pay no tax, which is different from $0.1x$. Therefore, $x > 150{,}000$, and under the second plan the first $150,000 of the President's yearly income would not be taxed, but the President would still pay tax equal to 16% of his yearly income over $150,000, i.e. $0.16(x - 150{,}000)$. Since the President's tax under either plan is the same, we get the following equation: $0.1x = 0.16(x - 150{,}000)$. Simplifying, we get $0.06x = 24{,}000$ $\Leftrightarrow x = 400{,}000$, which indeed greater than 150,000. Thus, the yearly income of the President of Dollarstan is $400,000. Note. The above solution assumes that the President of Dollarstan is a resident of Dollarstan, or at least pays income tax as a resident. **Answer: $400,000.**

3. In a given sequence, $a_1 = 2, a_2 = 3$, and the rule is: $a_{n+2} = a_{n+1}/\,a_n$ where $n = 1, 2, 3, \ldots$ Find a_{2025}.

Solution: Let's find the first 9 terms of this sequence: 2, 3, $\frac{3}{2}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, 2, 3, $\frac{3}{2}$, ….. Since the two consecutive terms a_7 and a_8 are the same as a_1 and a_2, accordingly, we conclude that the sequence repeats with a period of 6. $a_{2022} = a_6 = \frac{2}{3}$, and $a_{2025} = \frac{3}{2}$. **Answer:** $\frac{3}{2}$.

4. Two freight trains, each 1/6 mile long and traveling 60 miles per hour, meet and pass each other. How many seconds is it from the moment when the locomotives pass each other to the moment when the rear cars (the cabooses) pass each other?

Solution: When the locomotives meet, the cabooses are $\frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$ mile apart, and their net approaching speed is 120 miles per hour. It takes them $\frac{\frac{1}{3}}{120} = \frac{1}{360}$ hours = 10 seconds to meet. **Answer: 10 seconds.**

5. A square is drawn on each side of a triangle (each side of the triangle is a side of one of the squares). None of the four shapes overlap. The ratio of the areas of the squares is 1 : 2 : 3. What is the degree measure of the largest angle of the triangle?

Solution. Let a, b, c be the side lengths of the triangle, $a \leq b \leq c$. Then the areas of the squares described in the problem statement would be $a^2 \leq b^2 \leq c^2$. Since the ratio of the areas of the squares is 1 : 2 : 3, we get $a^2 : b^2 : c^2 = 1 : 2 : 3$, or $b^2 = 2a^2, c^2 = 3a^2$. Therefore, $a^2 + b^2 = a^2 + 2a^2 = 3a^2 = c^2$, so, the triangle is a right one by the converse of the Pythagorean Theorem. Since a right triangle contains one 90° angle and two acute angles, the largest angle of the triangle measures 90°. **Answer: 90°.**

6. It is known that in some math class, Al, Sam, and Mike are the same height. The average height of all students in the class except for Al is 63.1 inches. The average height of all students in the class except for both Al and Sam is 62 inches. The average height of all students in the class except for all three of them (Al, Sam, and Mike) is 60.625 inches. How many students are in this class?

Solution: This can be set up as a system of two linear equations in two unknowns, but since the number of students must be an integer it is easier to use the fraction denominators to make an educated guess. Since $63.1 = 63\frac{1}{10}$, and $60.625 = 60\frac{5}{8}$, one possibility is the following: there are 10 students in the class without Al, and 8 (precisely, 2 less than 10) students in the class without Al, Sam, and Mike. In this case, the total height of all students in the class except for Al, Sam, and Mike is $8 \cdot 60\frac{5}{8} = 485$ inches. The total height of all students in the class except for Al and Sam is $9 \cdot 62 = 558$ inches, therefore Mike is $558 - 485 = 73$ inches tall. The total height of all students in the class except for Al is $10 \cdot 63\frac{1}{10} = 631$inches, therefore, Sam is $631 - 558 = 73$ inches tall as well. Since this possibility satisfies all the conditions of the problem (there are no more restrictions on Al's height except that Al, Sam, and Mike have the same height), the answer is that there are $10 + 1 = 8 + 3 = 11$ students in the class. **Answer: 11.**

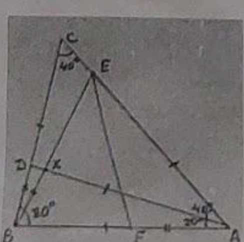
7. **Find the positive square root of the sum of the cubes of the first 15 natural numbers.**

Solution: The sum of the cubes of 1, ..., n is equal to the square of the sum of 1, ..., n. The proof could be found at https://mathschallenge.net/library/number/sum_of_cubes.Therefore, we only need to find the sum of 1, ... , n, which is $\frac{n(n+1)}{2} = \frac{15(16)}{2} = 120$. **Answer: 120.**

8. **Find real numbers a and b such that any pair (x, y) satisfying the equation $x/y = a$, satisfies the equation $(x+y)^2 = b$. Present as an ordered pair (a, b).**

Solution: Let a and b be the numbers that satisfy the above equations. The pairs $(a, 1)$ and $(2a, 2)$ satisfy the first equation. As for $(x+y)^2 = b$: $(a+1)^2 = (2a+2)^2 = b$; $(a+1)^2 = 4(a+1)^2$. It is possible only when $a = -1$ and $b = 0$. **Answer: (−1,0).**

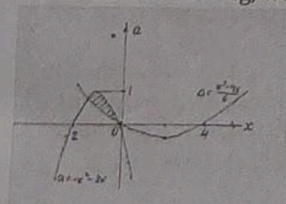
9. **In triangle ABC, $m\angle B = 80°$. Point D is located on $\overline{BC}$, and $\overline{AB} \cong \overline{AD} \cong \overline{CD}$. Point F is located on $\overline{AB}$, and $\overline{AF} \cong \overline{BD}$. Point E is located on $\overline{AC}$, and $\overline{AB} \cong \overline{AE}$. Find $m\angle AEF$.**



Solution: Since $\overline{AB} \cong \overline{AD}$, $m\angle ADB = m\angle ABD = 80°$ (Fig. 1). Since $\overline{AD} \cong \overline{CD}$, $m\angle DAC = m\angle DCA = 40°$. Thus, $m\angle BAE = 60°$, and triangle ABE is an equilateral triangle. $\overline{AD}$ and $\overline{BE}$ intersect at X. Then $m\angle BXD = m\angle BAX + m\angle ABX = 60° + 20° = 80° = m\angle BDX$, and $\overline{BX} \cong \overline{BD}$. Triangles ABX and EAF are congruent by SAS. It follows that $m\angle AEF = \angle BAX = 20°$ by $CPCTC$. **Answer: 20°.**

10. **Find all possible values of a such that $\begin{cases} x^2 + 2x + a \le 0 \\ x^2 - 4x - 6a \le 0 \end{cases}$ has one real solution.**

Solution: For solving, we are going to use a graphical approach. Let's rewrite the original system:



$\begin{cases} a \le -x^2 - 2x \\ a \ge \frac{x^2 - 4x}{6} \end{cases}$. All real solutions (x, a) form a region shown in Fig. 2 as a shaded region. The graphical interpretation of the solution: horizontal lines through the shaded region must have only one common point of interception. We conclude that only lines $a = 0$ or $a = 1$ satisfy this requirement. **Answer: 0, 1.**