Please attempt the problems on your own before consulting the answer key!

1. **Find the sum of all 3-digit numbers divisible by 7.**

Answer: 70336

Solution: Let's start by determining what we need to find. We want all 3-digit numbers by 7, so let's determine which numbers are included. First, let's long divide 100 by 7 to get 14 with a remainder of 2. $14 * 7 = 98$, so the first three digit number divisible by seven will be one multiple higher, $15 * 7 = 105$. Now for the highest number, long divide 1000 by 7 to get 142 with a remainder of 6. Therefore, the highest number will be $142 * 7 = 994$. The sum of all 3-digit numbers divisible by 7 will be $105 + 112 + 119 + \cdots + 987 + 994$. Since this is an arithmetic series, we can use the fact that the sum of an arithmetic sequence is $s = n(\frac{a_1 + a_n}{2})$, where a is the sequence $(105, 112, 119, ..., 987, 994)$ and n is the number of terms. There are $n = 142 - 15 + 1 = 128$ terms, so we get:

$$s = 128(\frac{a_1 + a_{128}}{2}) = 128(\frac{105 + 994}{2}) = 64(1099) = 70336$$

Thus, our answer is 70336.

2. **If $f(x) = \frac{x+1}{x-1}$ and $f(a) = 5$, what is the value of $f(2a)$?**

Answer: 2

Solution: Since we know what $f(a)$ is, let's plug in a into $f(x)$ to solve for a:

- $f(x) = \frac{x+1}{x-1}$ | original equation
- $f(a) = \frac{a+1}{a-1}$ | plugging in $x = a$
- $5 = \frac{a+1}{a-1}$ | substituting $f(a)$ with 5 because we know $f(a) = 5$
- $5(a - 1) = a + 1$ | multiplying both sides by $a - 1$
- $5a - 5 = a + 1$ | expanding
- $4a = 6$ | bring a terms to one side
- $a = \frac{3}{2}$ | simplifying

Great, we now know what a is. Let's now find $f(2a)$:

- $f(x) = \frac{x+1}{x-1}$ | original equation
- $f(2a) = \frac{2a+1}{2a-1}$ | plugging in $x = 2a$
- $f(2a) = \frac{2(\frac{3}{2})+1}{2(\frac{3}{2})-1}$ | substituting in $a = \frac{3}{2}$
- $f(2a) = \frac{3+1}{3-1}$ | simplifying
- $f(2a) = 2$ | simplifying

Thus, our answer is 2.

3. **Drew is skiing faster than Victor but slower than Evan. They all simultaneously started from point A, moving in the same direction along a circular path and stopped when all three were at point B. During this time, Evan passed Victor 13 times (not including the final stop). Find the total number of overtakes.**

Answer: 25

Solution: This question is quite difficult. If you want to understand how to do it systematically/algebraically, please look at the answer key's solution (it's at the bottom). This is a problem where I would recommend reading the answer key for a better answer. I will be explaining my brute-force-esk solution. With that out of the way, let's get into it. The problem gives few restraints (such as where points A and B are), but the problem nonetheless implies there is only one solution (the total number of overtakes is certain), so let's create a mock scenario with mock numbers (as long as our situation complies with the constraints, we should get the same answer). We know they are on a circular path, but we don't know if they end on the same spot they started on. However the answer should work no matter where point B is, so let's take the liberty of saying it's at the same position as point A (the start and end are the same point). Now, if Victor passed Evan 13 times, that means he did 14 more laps than Evan (think about this intuitively). Let's say Evan did one lap. Then Victor completed 15 laps, and Victor is 15 times faster than Evan. To give some dimensions, let's say the track is 15 meters, and that Evan has a speed of 1m/s, meaning it takes him 15 seconds to do one lap. I will let 15 seconds be the amount of time every skier skis. Victor is going 15 times faster than Evan so he does 15m/s, so one lap per second. In 15 seconds, he does 15 laps. Now, moving on to Drew, the question tells us that he is going faster than Evan but slower than Victor, but doesn't tell us his exact speed. Let's say he's going at 5m/s. In 15 seconds, he will cover 75m, an equivalent of $\frac{75}{15} = 5$ laps. Now realize that for Evan to do $k = 15 - 5 = 10$ more laps than Drew, he must've overtaken him $k - 1$ times. Try visualizing the following scenarios to see this more clearly:

Drew's lap count	Evan's lap count	Lap difference	Overtakes
1	2	1	0
2	3	1	0
1	3	2	1
2	4	2	1
1	4	3	2

Just from these initial few test scenarios we can see that overtakes is always lap difference $- 1$. If we did more tests, this would continue to hold true. Moving on, since Evan completed $k = 10$ more laps than Drew, he overtook Drew $k - 1 = 9$ times. Now let's count the number of times Drew overtook Victor. Since Drew did $k = 5 - 1 = 4$ more laps than Victor, he overtook Victor $k - 1 = 3$ times. Adding up the total number of overtakes we get $13 + 9 + 3 = 25$. Thus, our answer is 25.

4. **It is known that $a_1, a_2, \dots, a_n$ form an arithmetic sequence and $a_3 + a_9 = 8$. Find $a_1 + a_2 + \dots + a_{11}$.**

Answer: 44
Solution: There are two simple ways to solve this. They both take around the same time.
Algebraic method:
We know that $a_3 + a_9 = 8$, and because consecutive terms in an arithmetic sequence have a common difference, d, we can say that $a_3 = a_1 + 2d$ (a_3 is two terms from a_1). Likewise, we can also say $a_9 = a_1 + 8d$. Therefore:

$$a_3 + a_9 = a_1 + 2d + a_1 + 8d = 2a_1 + 10d = 8$$

We are solving for the sum from a_1 to a_{11} of an arithmetic sequence, so we can use the sum of an arithmetic sequence formula $s = n(\frac{a_1 + a_n}{2})$, where a is the sequence $(a_1, a_2, ..., a_{11})$ and n is the number of terms. There are $n = 11 - 1 + 1 = 11$ terms. Now let's do some algebra:

- $s = n(\frac{a_1 + a_n}{2})$ | original equation
- $s = 11(\frac{a_1 + a_{11}}{2})$ | plugging in $n = 11$ (there are 11 terms)
- $s = 11(\frac{a_1 + a_1 + 10d}{2})$ | substituting $a_{11} = a_1 + 10d$ (a_{11} is ten terms from a_1)
- $s = 11(\frac{2a_1 + 10d}{2})$ | simplifying
- $s = 11(\frac{8}{2})$ | substituting $2a_1 + 10d = 8$
- $s = 44$ | simplifying

Thus, our answer is 44.

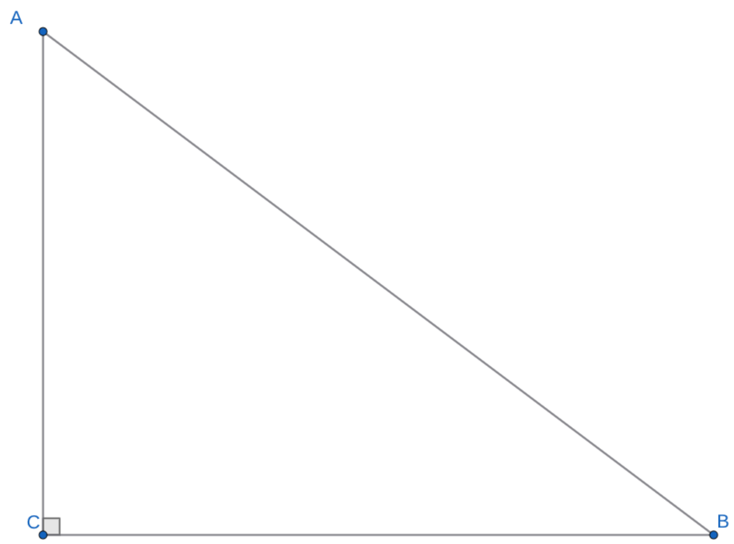
<u>Brute-force method:</u>
Let's try to create an arithmetic sequence in which $a_3 + a_9 = 8$. After some tinkering, I found a sequence that modelled the scenario: $a_n = n - 2$ ($a_3 + a_9 = 1 + 7 = 8$). From here, the sum of an arithmetic sequence is $s = n(\frac{a_1 + a_n}{2})$ so:

- $s = n(\frac{a_1 + a_n}{2})$ | original equation
- $s = 11(\frac{a_1 + a_{11}}{2})$ | plugging in $n = 11$ (there are 11 terms)
- $s = 11(\frac{-1 + 9}{2})$ | plugging in $a_1 = 1 - 2 =- 1$ and $a_{11} = 11 - 2 = 9$
- $s = 44$ | simplifying.

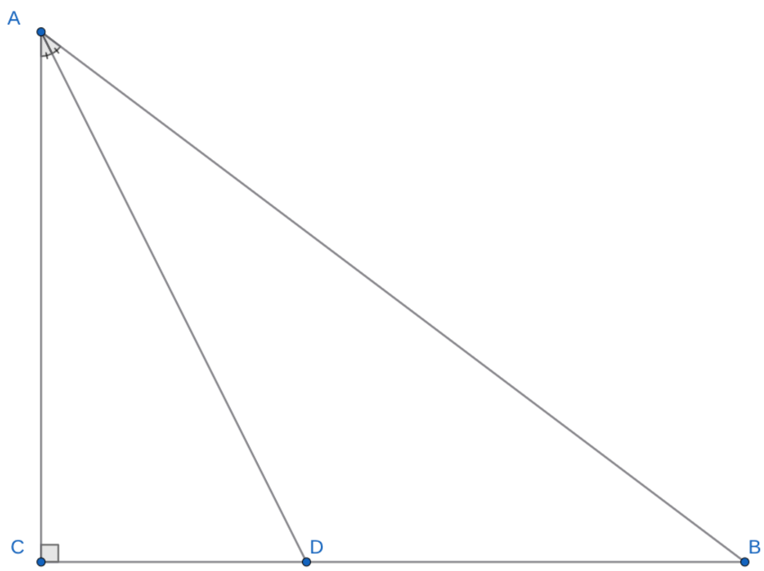
Thus, our answer is 44.

5. **In a right triangle ABC where angle C is a right angle, the angle bisector of angle A intersects the leg $\overline{CB}$ at point D. A segment that connects point D with a point of intersection of the medians, is perpendicular to $\overline{CB}$. Find the measure of angle B.**

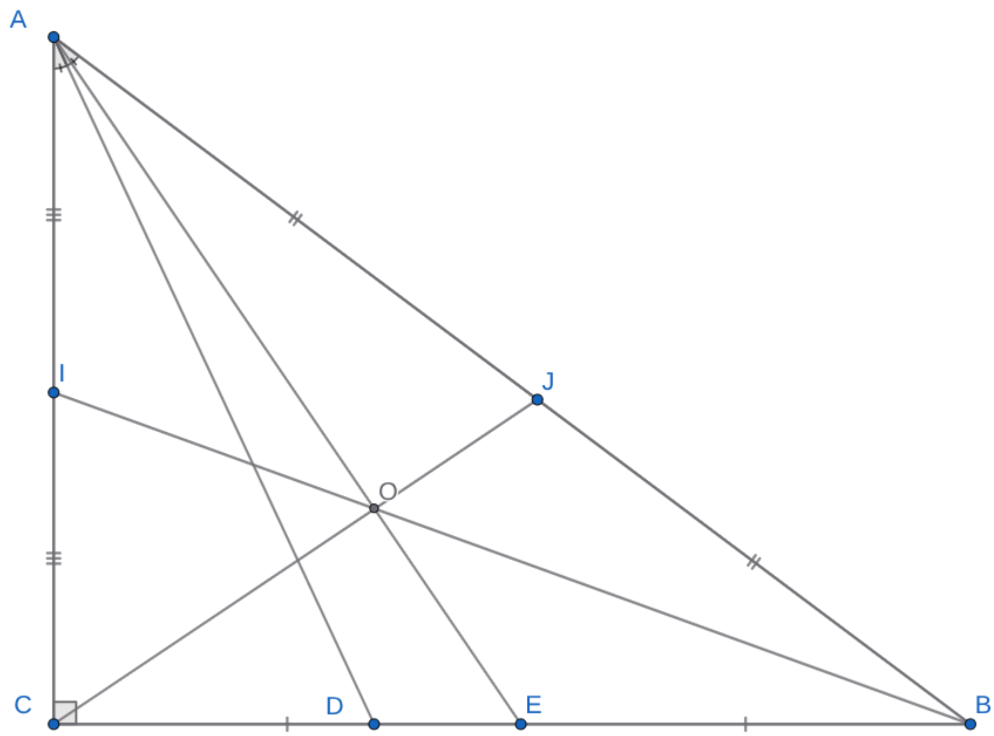
Answer: $30°$
Solution: Let's begin by drawing a diagram. When solving a geometry question, if I'm working with a shape I generally try to keep one of its sides horizontal, but that is just a suggestion. Anyways, here's a quick diagram of a right triangle (with angle C a right angle) that I sketched while solving:



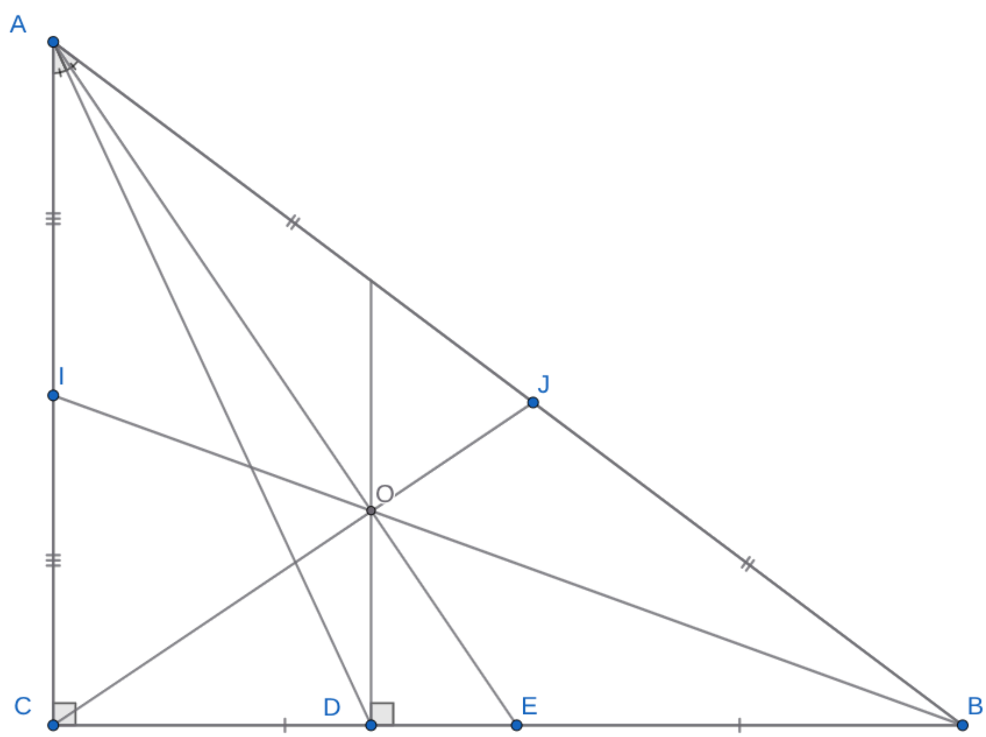
Now let's draw the angle bisector of angle A. It's a rough sketch, but let's try to split angle A into two congruent angles and mark point D as the intersection of the angle bisector and $\overline{CB}$:



Remember that in an angle bisector, according to the angle bisector theorem, the ratio of the non-shared sides is equal to the ratio of two segments created by splitting the opposite side (in this case $\frac{\overline{AC}}{\overline{AB}} = \frac{\overline{CD}}{\overline{DB}}$ which equals $\frac{\overline{AC}}{\overline{CD}} = \frac{\overline{AB}}{\overline{DB}}$). Moving on, the median of a triangle joins a vertex and to the midpoint of the other side (it bisects the other side). A point of intersection of the medians is the centroid of the triangle, and it splits each median into two segments of ratio $2{:}\,1$ (the vertex to centroid segment is the larger one). I'll sketch the three medians and their intersection point, point O:



Now that we have the intersection of the medians, the centroid, graphed, let's create a segment joining point D and the centroid which is perpendicular to $\overline{CB}$. During the competition, if you didn't put the centroid directly above point D you may have to move it so that it is because if the centroid is directly above point D, it will create a vertical segment which is perpendicular to the horizontal segment $\overline{CB}$. Let's graph it:



Now, let's solve. Earlier we found that, according to the angle bisector theorem, $\frac{\overline{AC}}{\overline{CD}} = \frac{\overline{AB}}{\overline{DB}}$. We can see that $\Delta BCI \sim \Delta BCO$ by AA similarity because both triangles share $\angle B$ and have a right angle. Because $\overline{BI}$ is a median, we know that $\overline{BO}:\overline{OI} = 2:1$ (or $\frac{\overline{BO}}{\overline{OI}} = \frac{2}{1} \Rightarrow \frac{1}{2}\overline{BO} = \overline{OI}$), and by extension:

$$\overline{BO} : \overline{BI} = \overline{BO} : \overline{BO} + \overline{OI} = \overline{BO} : \overline{BO} + \frac{1}{2}\overline{BO} = \overline{BO} : \frac{3}{2}\overline{BO} = 1 : \frac{3}{2} = 2 : 3$$

Because ΔBCI and ΔBCO are similar, all their side lengths have the same ratio, meaning that the same $2:3$ also holds for $\overline{BD} : \overline{BC}$. Therefore, $\frac{\overline{BD}}{\overline{BC}} = \frac{2}{3}$ and $\overline{BD} = \frac{2}{3}\overline{BC}$. Let's substitute that into the earlier equation we found, $\frac{\overline{AC}}{\overline{CD}} = \frac{\overline{AB}}{\overline{DB}}$, and solve:

- $\frac{\overline{AC}}{\overline{CD}} = \frac{\overline{AB}}{\overline{DB}}$ | original equation
- $\frac{\overline{AC}}{\overline{BC} - \overline{DB}} = \frac{\overline{AB}}{\overline{DB}}$ | substituting $\overline{CD} = \overline{BC} - \overline{BD}$
- $\frac{\overline{AC}}{\overline{BC} - \frac{2}{3}\overline{BC}} = \frac{\overline{AB}}{\frac{2}{3}\overline{BC}}$ | substituting $\overline{DB} = \frac{2}{3}\overline{BC}$
- $\frac{\overline{AC}}{\frac{1}{3}\overline{BC}} = \frac{\overline{AB}}{\frac{2}{3}\overline{BC}}$ | simplifying
- $\frac{\overline{AC}}{1} = \frac{\overline{AB}}{2}$ | multiplying both sides by $3\overline{BC}$
- $2\overline{AC} = \overline{AB}$ | simplifying

Great, now we know that $\overline{AC}$ is twice the length of $\overline{AB}$. Since ΔABC is a right triangle, and the hypotenuse is twice the length of one of the sides, ΔABC is a $30 - 60 - 90$ triangle. In a $30 - 60 - 90$ triangle, the side opposite the small side, $\overline{AC}$, is $30°$. Therefore, $m\angle B = 30°$. Another way of seeing this is:

$$m\angle B = sin(\frac{\overline{AC}}{\overline{AB}}) = sin(\frac{1}{2}) = 30°$$

Thus, our answer is $30°$.

6. **Two different integer numbers m and n are such that $(\frac{1}{m} - 2)$ and $(\frac{1}{n} - 2)$ are integer solutions of $x^2 + ax + b = 0$ with integer coefficients a and b. Find all possible values of $a + b$.**

Answer: 7

Solution: This question may seem difficult at first, but it is deceivingly easy. We are told that m and n are integers, and that $(\frac{1}{m} - 2)$ and $(\frac{1}{n} - 2)$ are integer solutions of a quadratic equation. The key here is that $(\frac{1}{m} - 2)$ and $(\frac{1}{n} - 2)$ are integers. In order for $(\frac{1}{x} - 2)$, where x is an integer, to be an integer, x must be either 1 or -1. Anything else and $(\frac{1}{x} - 2)$ will not be an integer. Thus, both m and n must be either 1 or -1, so the solutions to the quadratic equation should be either:

$$(\frac{1}{1} - 2) = (1 - 2) =- 1 \text{ or } (\frac{1}{-1} - 2) = (-1 - 2) =- 3$$

Since there are only two options for solutions to the quadratic equation, there are only four possibilities for the quadratic equation:

- $(x + 1)(x + 3)$
- $(x + 3)(x + 1)$
- $(x + 1)(x + 1)$
- $(x + 3)(x + 3)$

However, because the solutions are different integers, there are only two possible quadratic equations:

- $(x + 1)(x + 3)$

- $(x + 3)(x + 1)$

Expanding the expressions we get:

- $x^2 + 4x + 3$
- $x^2 + 4x + 3$

Although we could've noticed it earlier, now we can clearly see that the quadratic equations are the same, meaning that we actually have only 1 distinct (not the same) quadratic equation. Now just add:

$$a + b = 4 + 3 = 7$$

Thus, our answer is 7.

7. **Solve for** x: $3(log_2(sin\, x))^2 + log_2(1 - cos\, 2x) = 2$ **for** $0 \le x < 2\pi$**.**

Answer: $\frac{\pi}{6}, \frac{5\pi}{6}$

Solution: Note that this question is locked behind algebra 2 knowledge. Moving on, in the equation we can see a term that includes x which is squadred, a term with x which isn't squared, and a constant term. This is starting to seem like a quadratic equation, but let's make sure. Let's bring all the terms to one side and set $u = log_2 sinx$ and see what we can do:

- $3(log_2(sin\, x))^2 + log_2(1 - cos\, 2x) = 2$ | original equation
- $3u^2 + log_2(1 - cos2x) - 2 = 0$ | substituting $u = log_2 sin(x)$ and moving 2 to the left side

Now, if we just had u in the second term we would have a quadratic that we could solve. Let's see if we can make that happen. From our [trig identities](), a double angle identity for $cos(2x)$ that includes $sinx$ is $cos(2x) = 1 - 2sin^2 x$, so:

$$log_2(1 - cos2x) = log_2(1 - (1 - 2sin^2 x)) = log_2(2sin^2 x)$$

Now let's do some rearranging to get just $log_2 sinx$:

- $log_2(2sin^2 x)$ | original equation
- $log_2(sin^2 x) + log_2(2)$ | [product rule for logs]()
- $2log_2(sinx) + log_2(2)$ | [power rule for logs]()
- $2log_2(sinx) + 1$ | [identity rule for logs]()

Ok, now that we know that $log_2(1 - cos2x) = 2log_2(sinx) + 1$, let's substitute that back into the equation from earlier:

- $3u^2 + log_2(1 - cos2x) - 2 = 0$ | original equation
- $3u^2 + 2log_2(sinx) + 1 - 2 = 0$ | substituting $log_2(1 - cos2x) = 2log_2(sinx) + 1$
- $3u^2 + 2log_2(sinx) - 1 = 0$ | simplifying
- $3u^2 + 2u - 1 = 0$ | substituting $u = log_2 sinx$

Great, now we have a quadratic equation. Let's pray it's factorable… it is!

$$3u^2 + 2u - 1 \equiv (3u - 1)(u + 1)$$

So either $3u - 1 = 0$ or $u + 1 = 0$. Let's explore these paths, keeping in mind that $0 \le x < 2\pi$:

- $3u - 1 = 0$ | original equation
- $3u = 1$ | bring u terms to one side
- $u = \frac{1}{3}$ | solving for u

- $log_2 sinx = \frac{1}{3}$ | substituting $u = log_2 sinx$
- $2^{\frac{1}{3}} = sinx$ | exponentiation property of logarithms
- $\sqrt[3]{2} = sinx$ | rewriting

Because $\sqrt[3]{2} > 1$ while the range of $sinx$ is $[-1, 1]$, there is no solution. Let's move on to $(u + 1)$:

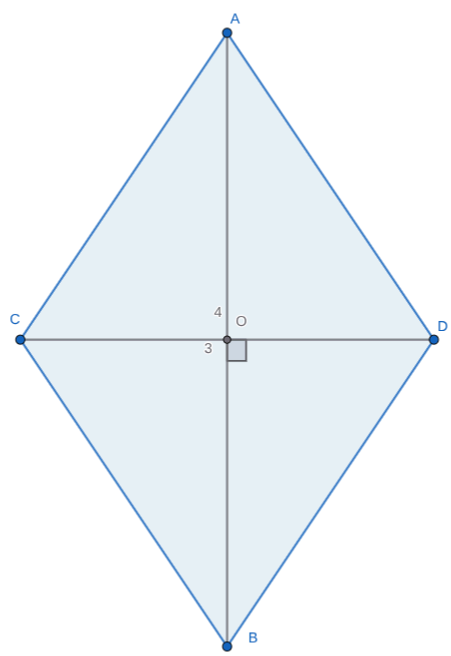
- $u + 1 = 0$ | original equation
- $u = -1$ | solving for u
- $log_2 sinx = -1$ | substituting $u = log_2 sinx$
- $2^{-1} = sinx$ | exponentiation property of logarithms
- $\frac{1}{2} = sinx$ | rewriting

On the interval $0 \leq x < 2\pi$, $sinx = \frac{1}{2}$ at $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ (feel free to quickly plug these answers back into the problem during the competition to make sure you got it right). Thus, our answer is $\frac{\pi}{6}, \frac{5\pi}{6}$.

8. **The lengths of the diagonals of a rhombus are in a ratio $3:4$. How many times is the area of a rhombus larger than the area of the circle inscribed in it? Express as a ratio.**

Answer: $\frac{25}{6\pi} : 1$

Solution: There are two main ways to go about solving this question. I prefer giving definite dimensions to the rhombus, but leaving the side lengths in variable form (which is what the answer key does) is also viable (and it will make harder questions easier). I will go through a solution in which I assign definite dimensions to the rhombus, but feel free to check out the answer key's solution which keeps all values in variable form. Note that the basic definition of a rhombus is a polygon with four congruent sides where opposite sides are parallel. Moving on, we know the ratio of the length of the diagonals in the rhombus is $3:4$. The question then asks us to find the ratio of the area of the rhombus to the area of the circle inscribed in it, so there aren't any more specifications other than the diagonals are in a $3:4$ ratio. Let's say that the lengths of the diagonals are 3 and 4 respectively ($\overline{AB} = 4$ and $\overline{CD} = 3$):



Knowing the lengths of the diagonals, the area of the rhombus can be found using the formula for the area of a rhombus where d_1 and d_2 are the lengths of the diagonals of the rhombus:

$$A_{rhombus} = \frac{d_1 * d_2}{2} = \frac{3 * 4}{2} = 6$$

Great, now that we have the area of the rhombus, let's try and get the area of the circle. For that, let's try and find the radius of the circle. The intersection of the diagonals of the rhombus (aka the incenter) splits both diagonals in half, so:

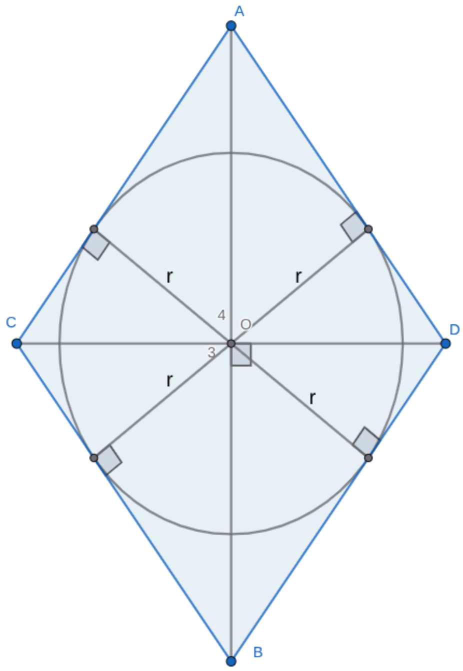
$$\overline{CO} = \overline{OD} = \frac{3}{2} \text{ and } \overline{AO} = \overline{OB} = 2$$

Additionally, let's also calculate the side lengths of the rhombus. By the Pythagorean Theorem, we can calculate:

$$\overline{CA} = \sqrt{(\overline{CO})^2 + (\overline{AO})^2} = \sqrt{(\frac{3}{2})^2 + 2^2} = \sqrt{\frac{9}{4} + \frac{16}{4}} = \sqrt{\frac{25}{4}} = \frac{5}{2}$$

Since all side lengths in a rhombus are equal, the lengths of all the other sides of the rhombus are also $\frac{5}{2}$.

Moving on, the radius of the inscribed circle will be the height, dropped from point O, of any of the four triangles congruent triangles ($\Delta DOA, \Delta AOC, \Delta COB, \Delta BOD$), as shown below:



If we find the radius of the circle, we will be able to find its area. Using the fact that the area of a parallelogram (which we know is 6) is $b * h$, we can solve for r:

$$6 = b * h = \overline{BD} * (r + r) = \frac{5}{2} * 2r = 5r,\ r = \frac{6}{5}$$

Knowing the radius, we can find the area of the circle:

$$A_{circle} = \pi r^2 = \pi(\frac{6}{5})^2 = \frac{36\pi}{25}$$

Now that we know the area of both the rhombus and the circle inscribed inside of it, we can find how many times larger the area of the rhombus is than the circle inscribed inside of it by dividing the area of the rhombus, $A_{rhombus}$, by the area of the circle, A_{circle}:

$$\frac{A_{rhombus}}{A_{circle}} = \frac{6}{\frac{36\pi}{25}} = \frac{150}{36\pi} = \frac{25}{6\pi}$$

Thus, the ratio of the area of the rhombus to the area of the circle inscribed inside it is our answer $\frac{25}{6\pi} : 1$.

9. **If $f(x) = mx + b$ for $m < 0$ and $f(x) = 4f^{-1}(x) + 3$, what is $m + b$?**

Answer: -5

Solution: We are told that $f(x) = mx + b$ (where m and b are constants) and that $m < 0$. Then, we are told that $f(x) = 4f^{-1}(x) + 3$ ($f^{-1}(x)$ is notation for the inverse of $f(x)$). Let's start by finding what $f^{-1}(x)$, the inverse of $f(x)$, is. To find $f^{-1}(x)$, switch $f(x)$ and x in the original equation, change $f(x)$ to $f^{-1}(x)$, and solve for $f^{-1}(x)$:

- $f(x) = mx + b$ | original equation
- $x = mf(x) + b$ | swap $f(x)$ and x
- $x = mf^{-1}(x) + b$ | change $f(x)$ to $f^{-1}(x)$
- $x - b = mf^{-1}(x)$ | solving for $f^{-1}(x)$
- $\frac{x-b}{m} = f^{-1}(x)$ | solving for $f^{-1}(x)$

Great, now we know that $f^{-1}(x) = \frac{x-b}{m}$. Moving on, we know that both $f(x)$ and $f^{-1}(x)$ must be defined for all x, so any value of x we input must work. Since we have two variables, let's plug in two values for x, 1 and 0 (because they are easy to do algebra with), to create a system of two equations:

$$\begin{cases} f(0) = 4f^{-1}(0) + 3 \\ f(1) = 4f^{-1}(1) + 3 \end{cases}$$

Let's expand both of them:

- $f(0) = 4f^{-1}(0) + 3$ | original equation
- $m(0) + b = 4(\frac{0-b}{m}) + 3$ | expanding $f(0)$ and $f^{-1}(0)$
- $b = \frac{-4b}{m} + 3$ | simplifying

- $f(1) = 4f^{-1}(1) + 3$ | original equation
- $m(1) + b = 4(\frac{1-b}{m}) + 3$ | expanding $f(1)$ and $f^{-1}(1)$
- $m + b = \frac{4-4b}{m} + 3$ | simplifying

Seeing as both of the equations have $\frac{-4b}{m}$ (remember $\frac{4-4b}{m} = \frac{4}{m} + \frac{-4b}{m}$), if we substitute b from the first equation into the second, we should be able to eliminate b:

- $m + (\frac{-4b}{m} + 3) = \frac{4-4b}{m} + 3$ | substituting $b = \frac{-4b}{m} + 3$ into $m + b = \frac{4-4b}{m} + 3$
- $m + \frac{-4b}{m} + 3 = \frac{4}{m} + \frac{-4b}{m} + 3$ | intermediary step
- $m = \frac{4}{m}$ | cancelling out terms
- $m^2 = 4$ | bring m to one side
- $m = \pm 2$ | solving for m

Now we know m is either 2 or -2. Remember that the problem originally told us that $m < 0$, so $m = -2$ is the only solution. Using m, let's now let's solve for b (using any prior equation with m and b):

- $b = \frac{-4b}{m} + 3$ | original equation
- $b = \frac{-4b}{-2} + 3$ | substituting $m = -2$
- $b = 2b + 3$ | simplifying

- $-b = 3$ | simplifying
- $b = -3$ | simplifying

Now that we have both m and b, we can calculate $m + b = -2 + -3 = -5$. Thus, our answer is -5.

10. **Find all values of b such that $\frac{x^2 - (3b-1)x + 2b^2 - 2}{x^2 - 3x - 4} = 0$ has exactly one real solution.**

Answer: $\frac{1}{2}, -2$

Solution: Ok, this is a relatively tough one. To begin, in order for $\frac{x^2 - (3b-1)x + 2b^2 - 2}{x^2 - 3x - 4}$ to be 0, the numerator (top part of the fraction) must be 0 while the bottom is not 0 (dividing by 0 is undefined). To begin, we can factor the bottom:

$$x^2 - 3x - 4 = (x - 4)(x + 1)$$

Therefore, $x \neq -1$ and $x \neq 4$. Now, for the numerator, let's try using the quadratic formula (note that in $x^2 - (3b - 1)x + 2b^2 - 2$, $a = 1, b = -3b + 1, c = 2b^2 - 2$) to find the solutions to the quadratic:

- $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ | original quadratic formula
- $x = \frac{-(-3b+1) \pm \sqrt{(-3b+1)^2 - 4(1)(2b^2 - 2)}}{2(1)}$ | plugging in $a = 1, b = -3b + 1, c = 2b^2 - 2$
- $x = \frac{3b - 1 \pm \sqrt{9b^2 - 6b + 1 - 8b^2 + 8)}}{2}$ | expanding
- $x = \frac{3b - 1 \pm \sqrt{b^2 - 6b + 9}}{2}$ | combining like terms
- $x = \frac{3b - 1 \pm \sqrt{(b-3)^2}}{2}$ | factoring
- $x = \frac{3b - 1 \pm (b-3)}{2}$ | simplifying
- $x = 2b - 2, b + 1$ | solving

Great, now we know that the solutions to the quadratic equation are $x = 2b - 2$ and $x = b + 1$. Now let's think about what we're trying to find. We want a value of b such that the equation $\frac{x^2 - (3b-1)x + 2b^2 - 2}{x^2 - 3x - 4} = 0$ has only one solution. We already found that for any b, the numerator has two solutions (which may or may not be distinct). However if one of those solutions is $x = -1$ or $x = 4$, it will not actually be a solution to the equation because of division by 0. So, if we set $2b - 2$ and $b + 1$ equal to -1 and 4 we will find values of b for which at least one of the possible solutions are removed. Let's do that now:

- $-1 = 2b - 2, 1 = 2b, b = \frac{1}{2}$
- $4 = 2b - 2, 6 = 2b, b = 3$
- $-1 = b + 1, b = -2$
- $4 = b + 1, b = 3$

Let's interpret these solutions. First let's look at the two non-repeating solutions, $b = \frac{1}{2}$ and $b = 3$. They are values of b for which -1 is one of the solutions to the numerator, but since -1 is also a solution to the denominator, it will not be a solution to the equation. The quadratic's other solution will, however, still be a solution to the numerator (we know it isn't 4 because the only time 4 is a solution to the numerator is when $b = 3$). Now, we got $b = 3$ twice, which means that the solution to the numerator when $b = 3$ is 4 (it is a repeated solution because the quadratic has a vertex on the x-axis). Because 4 is also a solution to the denominator, it will not be a solution to $\frac{x^2 - (3b-1)x + 2b^2 - 2}{x^2 - 3x - 4} = 0$ when $b = 3$. Therefore, $b = 3$ will not work. You can see this visually here (note that when $b = 3$, there is a hole at $x = 4$ which may not be apparent on visual inspection). Thus, our only two remaining solutions are our answer: $\frac{1}{2}, -2$.

Competition Answer Key:

Central Jersey Mathematics League Contest

[Solutions]

December 10, 2025

1. Find the sum of all 3-digit numbers divisible by 7.

Solution: The first such number is 105 since the closest 2-digit number divisible by 7 is 98. The last such number is 994. All such numbers form an arithmetic sequence with a common difference of 7. Let the number of terms is the sequence is n. Then $105 + (n-1) \cdot 7 = 994$. From here, $n = 128$, and the sum is $\frac{105 + 994}{2} \cdot 128$ $= 70336$. **Answer: 70336.**

2. If $f(x) = \frac{x+1}{x-1}$ and $f(a) = 5$, what is the value of $f(2a)$?

Solution: Solving for $x = a$, $\frac{a+1}{a-1} = 5$, and $a = \frac{3}{2}$, $2a = 3$, $f(2a) = f(3) = 2$. **Answer: 2.**

3. Drew is skiing faster than Victor but slower than Evan. They all simultaneously started from point A, moving in the same direction along a circular path and stopped when all three were at point B. During this time, Evan passed Victor 13 times (not including the final stop). Find the total number of overtakes.

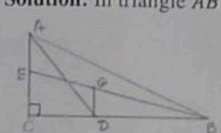
Solution: These are 13 moments of time when Evan overtook Victor; break the entire movement time into 14 intervals, and for each such interval Evan outperformed Victor exactly by one lap. That means that Evan made 14 laps more than Victor. Let's assume that Drew has made k laps more than Victor, where $0 < k < 14$. Similarly, we get that Drew overtook Victor $(k-1)$ times. But Drew did $(14-k)$ laps less than Evan, and therefore, Evan overtook him $(13-k)$ times. In total, the overtaking occurred $13 + (k-1) + (13-k) = 25$ times. **Answer: 25.**

4. It is known that $a_1, a_2, \ldots, a_n$ form an arithmetic sequence and $a_3 + a_9 = 8$. Find $a_1 + a_2 + \cdots + a_{11}$.

Solution: For this sequence, we have, $a_3 = a_1 + 2d$, $a_9 = a_1 + 8d$, $a_{11} = a_1 + 10d$, where d is the common difference. It is known that $a_3 + a_9 = 8$ or $a_1 + 2d + a_1 + 8d = 2a_1 + 10d = a_1 + (a_1 + 10d) = 8$. We need to find the sum of eleven terms: $\frac{a_1 + a_{11}}{2} \cdot 11 = \frac{2a_1 + 10d}{2} \cdot 11 = 44$. **Answer: 44.**

5. In a right triangle ABC where angle C is a right angle, the angle bisector of angle A intersects the leg $\overline{CB}$ at point D. A segment that connects point D with a point of intersection of the medians, is perpendicular to $\overline{CB}$. Find the measure of angle B.

Solution: In triangle ABC (Fig. 1), $\overline{BE}$ represents a median, G represents a point of intersection of the medians (a Centroid or Center of Gravity). It is given, that $\overline{GD} \perp \overline{CB}$. According to properties of the medians in a triangle, $EG : GB = 1 : 2$. Since $\overline{GD} \parallel \overline{EC}$, triangles ECB and GDB are similar, and $CD : DB = EG : GB = 1 : 2$. Using the property of an angle bisector in a triangle, $CD : DB = AC : AB = 1 : 2 = \sin B = \frac{1}{2}$. Thus, $m\angle B = 30°$. **Answer: 30°.**



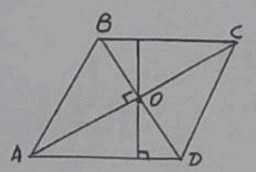
6. Two different integer numbers m and n are such that $(\frac{1}{m} - 2)$ and $(\frac{1}{n} - 2)$ are integer solutions of $x^2 + ax + b = 0$ with integer coefficients a and b. Find all possible values of $a + b$.

Solution: By Vieta theorem, $(\frac{1}{m} - 2) + (\frac{1}{n} - 2) = -a$; $(\frac{1}{m} - 2) \cdot (\frac{1}{n} - 2) = b$, where a and b are integers. Since $\frac{1}{m} - 2$ is an integer, m must be either 1 or -1. By analogy, for $\frac{1}{n} - 2$ to be an integer, n must be either 1 or -1. Since m and n are different integers, consider two cases: $m = 1, n = -1$, or $m = -1, n = 1$. It follows that -3 and -1 are the solutions of the given quadratic equation; $a = 4$, $b = 3$, and $a + b = 7$. **Answer: 7.**

7. Solve for x: $3(\log_2(\sin x))^2 + \log_2(1 - \cos 2x) = 2$ for $0 \le x < 2\pi$.

Solution: We have, $\log_2(1 - \cos 2x) = \log_2(2\sin^2 x) = 1 + 2\log_2(\sin x)$. Thus, the given equation is defined for all x where $\sin x > 0$. Let $\log_2(\sin x) = t$. Then we get $3t^2 + 2t - 1 = 0$, and after solving for t, $t = 1$, $\frac{1}{3}$. It follows that $\sin x = \frac{1}{2}$, $2^{\frac{1}{3}}$. Since $2^{\frac{1}{3}} > 2^0 = 1$, only $\sin x = \frac{1}{2}$ leads to a solution. **Answer:** $\frac{\pi}{6}$, $\frac{5\pi}{6}$.

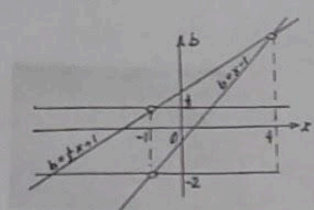
8. The lengths of the diagonals of a rhombus are in a ratio 3 : 4. How many times is the area of a rhombus larger than the area of the circle inscribed in it? Express as a ratio.



Solution: In the rhombus $ABCD$, (Fig. 2), let the side be a. The circle with center O is inscribed. Since the diagonals of the rhombus are perpendicular, the triangle AOB is a right triangle, and each of its legs is equal to one-half of corresponding diagonal. Let $AO = 4x$ and $DO = 3x$, then $(4x)^2 + (3x)^2 = a^2$. Solving for x, we obtain, $x = \frac{1}{5}a$. Thus, the diagonals of the rhombus are $\frac{6}{5}a$ and $\frac{8}{5}a$, and its area $A = \frac{1}{2} \cdot \frac{6}{5}a \cdot \frac{8}{5}a = \frac{24}{25}a^2$. The radius of the inscribed circle is equal to one-half of the rhombus's height, which could be found using the formula for the area of the trapezoid as $h = \frac{A}{a} = \frac{24}{25}a$. Thus, the radius of the inscribed circle is $\frac{12}{25}a$, and the area of the inscribed circle is $\frac{144}{625}\pi a^2$. Then the ratio of the area of the rhombus to the area of the circle inscribed in it is $25 : 6\pi$ or $\frac{25}{6\pi} : 1$.
Answer: $\frac{25}{6\pi} : 1$.

9. If $f(x) = mx + b$ for $m < 0$ and $f(x) = 4f^{-1}(x) + 3$, what is $m + b$?
Solution: First, $f^{-1}(x) = \frac{x-b}{m}$. Then $mx + b = \frac{4x-4b}{m} + 3$. This must be true for all x, so let $x = 0$ and $x = 1$, thereby obtaining the system: $\begin{cases} b = \frac{-4b}{m} + 3 \\ m + b = \frac{4-4b}{m} + 3 \end{cases}$ or $\begin{cases} bm = -4b + 3m \\ m^2 + bm = 4 - 4b + 3m \end{cases}$. Substituting for bm, we have, $m^2 - 4b + 3m = 4 - 4b + 3m$, giving $m^2 = 4$, so $m = -2$ (must be negative). Then $b = -3$. The sum $m + b$ is: $-2 + (-3) = -5$. **Answer:** -5.

10. Find all values of b such that $\frac{x^2 - (3b-1)x + 2b^2 - 2}{x^2 - 3x - 4} = 0$ has exactly one real solution.



Solution: The given equation is equivalent to the following system: $\begin{cases} \begin{cases} x = 2b - 2 \\ x = b + 1 \end{cases} \\ x^2 - 3x - 4 \neq 0 \end{cases}$. From the last equation, $x \neq 4$ or $x \neq -1$. Let's graph two lines $b = \frac{1}{2}x + 1$ and $b = x - 1$ on the coordinate plane (Fig. 3). The points on the lines where b is not defined or where x-coordinate equals to -1, indicate that the above system has only one real solution (the last such point, where $x = 4$, is not in the domain). The fact that the above system has only one real solution could be verified by substituting the values of b into the given function. Thus, $b = -2$, $\frac{1}{2}$. **Answer:** -2, $\frac{1}{2}$.