1. **If k, m, and n are positive numbers, and k is m percent of n, what percent of m is nk?**

Answer: n^2

Solution: This problem is all about translating English into math. To begin, we know k is m percent of n. Knowing that x% of anything is just $\frac{x}{100}$ times that value, if k is m percent of n, then $k = \frac{m}{100} * n$. From here, knowing that some percent of m is nk, we construct the second equation: $kn = \frac{x}{100} * m$, where x the unknown. Now we have a system of two equations. Substituting k in the second equation with the first equation, we get:

- $(\frac{m}{100} * n)n = \frac{x}{100} * m$ | Original equation
- $\frac{mn^2}{100} = \frac{xm}{100}$ | Combining terms
- $n^2 = x$ | Multiplying both sides by $\frac{100}{m}$ to solve for x

Thus, m is n^2% of nk. To see this, below are some example values:

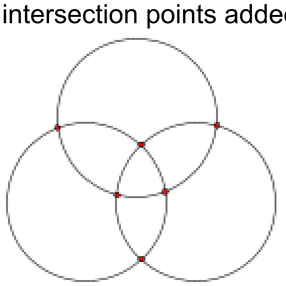
k	m	n	nk	n^2 (nk is n^2% of m)
20	100	20	400	400
10	10	100	1000	10000
5	$33\frac{1}{3}$	15	75	225
50	10	500	25000	250000
8	25	32	256	1024

During the competition, it may be worth it to check if your answer is correct using values for k, m, and n. However, more than one check is not really necessary. Nonetheless, we can see our answer is correct: n^2

2. **Find the maximum number of points of intersection among three circles and two lines in the same plane.**

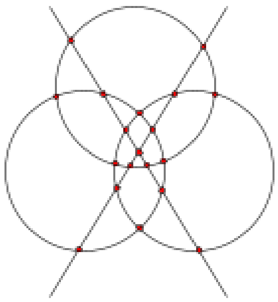
Answer: 19

Solution: We have three circles and two lines, and want to find the maximum number of intersections possible. Starting with the three circles, through some simple testing we find that the maximum amount of times two (non-overlapping) circles can intersect is 2. Adding a third third circle, the maximum number of intersection points added is 4, for a total of 6 intersections:



Now we add the two lines. The maximum number of intersections between a line and circle is two (try it out!) so the theoretical maximum number of additional intersections when a line is added is 2

intersections per line times 3 circles equals at most 6 additional intersections when we add one line. With some trial and error, we see that this is possible. So, with the additional line, we are up to 12 intersections. For the final line, we know that the maximum number of intersections with the circles is still 6 (because there are still six circles), but because of the first line, there may be one additional intersection (where the two lines intersect). With some moving around of the circles and lines, we are indeed able to able to add the six intersections with the circles and the intersection with the first line for an additional 7 intersections, leaving us with 19 intersections total. Below is a visual representation:



3. **There are 13 weights on the table, stacked in a row and ordered by their mass (the lightest is on the left, and the heaviest is on the right). It is known that the mass of each weight in grams represents an integer, the masses of two neighboring weights differ by no more than 5g, and the sum mass of all weights does not exceed 2025g. Find the maximum possible mass of the heaviest weight.**

Answer: 185

Solution: So we have 13 weights that differ by no more than 5g in weight and the sum of their weights is < 2025g. We want the maximum weight of the heaviest one, while ensuring the weights all have integer weights. Realize that to have the heaviest possible weight, we want the weights to differ in weight as much as possible. Think about it; if you have 10 weights that add up to less than 50g, four 12g weights have a lower maximum weight of 12g than, for example, four weights weighing 5g, 10g, 15g, and 20g respectively, which have a maximum weight of 20g. With that in mind, we want each weight to differ in weight by the maximum amount of 5g. Thus, we can create the following equation, where x is the weight of the first (left most) weight:

$$x + (x+5) + (x+10) + (x+15) + \dots + (x+60) < 2025$$

Now just solve for $x + 60$ (because we want the weight of the heaviest, not the lightest):

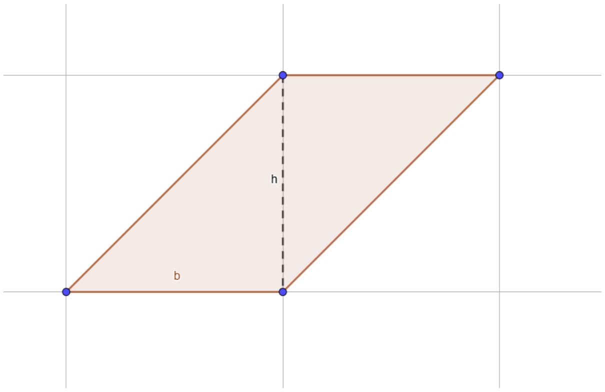
- $13x + 5 + 10 + 15 + \dots + 60 < 2025$ | grouped x terms
    - $5 + 10 + 15 + \dots + 60 = 12\frac{5+60}{2} = 390$ | sum of an arithmetic sequence formula
- $13x + 390 < 2025$ | substituting
- $13x < 1635$ | simplifying
- $x < 125 + \frac{10}{13}$ | long division
- $x = 125$ | we want the greatest integer x less than $125 + \frac{10}{13}$
- $x + 60 = 185$ | adding 60 to both sides to get the weight of the heaviest weight

Therefore, our answer is 185. Both 185 and 185g are acceptable answers.

4. **Four vertices of a square grid where each square is 1by1, have been chosen to form a parallelogram. It turns out that neither the parallelogram's sides nor its interior contain any other vertices of the grid, and it is known that the parallelogram is not a rectangle. Find the area of the parallelogram.**

Answer: 1

Solution: Sketch a coordinate grid. We know it's a parallelogram that doesn't have any vertices of the grid on its sides or inside of it (other than the four vertices of the parallelogram). With that information, we can create a sample parallelogram:



It is both not a rectangle and doesn't contain any vertices, satisfying the rules set forth by the question. Area of a parallelogram is $bh = 1 * 1 = 1$. Thus, our answer is 1.

Another possible solution uses Pick's Theorem. It gives us the formula

$$A = I + \frac{1}{2}B - 1$$

Where A is the area, I is number of lattice points (integer x,y coordinates) in the interior of the polygon and B is the number of lattice points on the sides of the polygon. Keep in mind that Pick's theorem is only applicable to polygons whose vertices are lattice points. For polygons with non-lattice point vertices, look to the Shoelace Theorem. Also note that there was no need to know Pick's Theorem, but knowing it would allow you to solve the question quicker and in turn grant you more time for other problems. For more information on Pick's Theorem, look here.

5. **Emma took the word $GEOMETRY$ and replaced each of its eight letters with a number corresponding to the code $A = 1, B = 2, \dots, Z = 26$. Then she multiplied six of these numbers to get a cube of an integer. Find this integer.**

Answer: 150

Solution: So each letter in geometry maps to a number and six of the eight numbers multiply to the cube of an integer. To start, let's replace each letter in $GEOMETRY$ with its corresponding number:

$$7\ 5\ 15\ 13\ 5\ 20\ 18\ 25$$

Now let's prime factorize each number so we can more easily see which numbers Emma used:

$$7^1,\ 5^1,\ 3^1 * 5^1, 13^1, 5^1, 2^2 * 5, 3^2 * 2^1, 5^2$$

Since we are looking for the cube of a number, we need a multiple of three of each of its prime factors (if you are unsure why, please ask us or look online!). Because there is only one 7 and only one 13, those cannot be used. Thus, we are left with:

$$5^1, 3^1 * 5^1, 5^1, 2^2 * 5, 3^2 * 2^1, 5^2$$

Multiplying all together (while keeping them prime factorized to make it easy) we get:

$$5^6 * 3^3 * 2^3$$

This is the cube of the integer. To find the integer, we cube root:

$$\sqrt[3]{5^6 * 3^3 * 2^3} = 5^2 * 3^1 * 2^1 = 25 * 6 = 150$$

Thus, our answer is 150.

6. **Calculate the sum**
$$1^2 + 2^2 - 3^2 - 4^2 + 5^2 + 6^2 - 7^2 - 8^2 + 9^2 + 10^2 - \cdots - 2023^2 - 2024^2 + 2025^2.$$

Answer: 2025

Solution: This is pattern recognition. Evaluating the first few terms we get:

$$1 + 4 - 9 - 16 + 25 + 36 - 49 - 64 + 81 + 100$$

Now try looking for a pattern. Try grouping together terms. After some playing around with numbers, you should realize that both the 2-5 and 6-9 terms add up to 4:

$$+ 4 - 9 - 16 + 25 =- 5 + 9 = 4$$

$$+ 36 - 49 - 64 + 81 =- 13 + 17 = 4$$

To see if the pattern continues, we do it with the next four terms ($+ 10^2 - 11^2 - 12^2 + 13^2$):

$$+ 100 - 121 - 144 + 169 =- 11 + 15 = 4$$

Thus, we can be fairly confident this pattern continues. From the second term (where the pattern starts) to the two thousand and twenty fifth term (where the pattern stops because the series stops) there are $2025 - 2 + 1 = 2024$ terms. Since every four terms adds to four, there are $\frac{2024}{4} = 506$ of the four term subseries, and since each of the four term sequences adds to 4, the sum of all of the 506 four term subseries will be $506 * 4 = 2024$. Adding the first term, 1, we get $1 + 2024 = 2025$. Our answer is 2025.

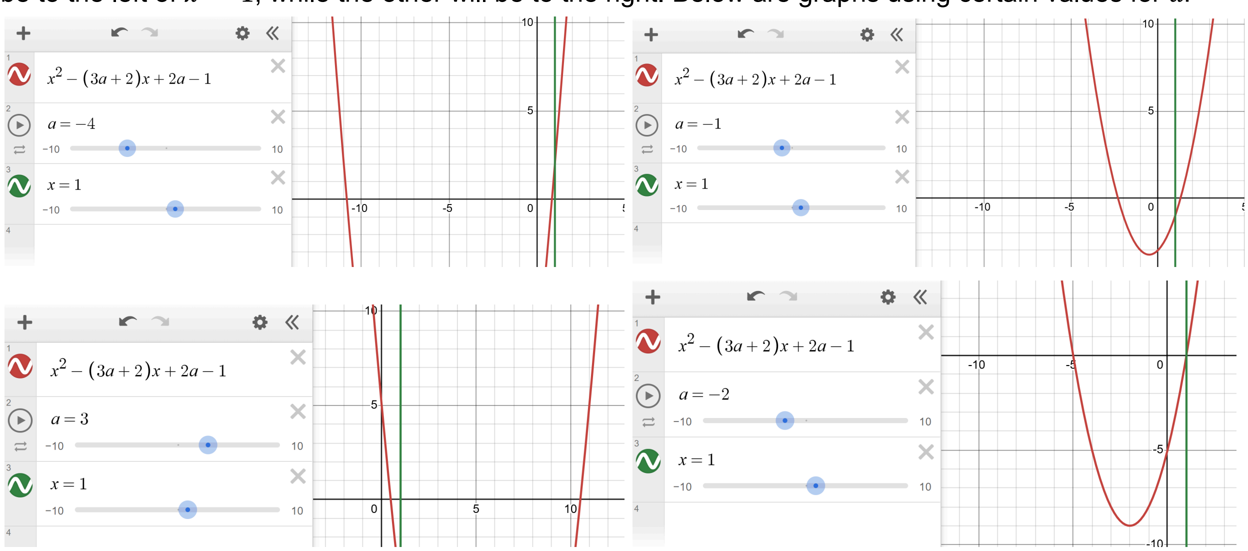
7. **Find real a such that one solution of $x^2 - (3a + 2)x + 2a - 1 = 0$ is greater than 1 and the other solution is less than 1?**

Answer: $a >- 2$

Solution: Realize that if one solution of the quadratic is less than 1, while the other is greater than 1, then the value at $x = 1$ must be < 0 (because the parabola is opening up). Therefore, we set the value of the quadratic at $x = 1$ to be less than 0 and then solve for a:

- $x^2 - (3a + 2)x + 2a - 1 < 0$ | original equation
- $1^2 - (3a + 2)(1) + 2a - 1 < 0$ | substituting in $x = 1$
- $1 - 3a - 2 + 2a - 1 < 0$ | expanding
- $- 2 - a < 0$ | simplifying
- $- a < 2$ | isolating a
- $a >- 2$ | dividing both sides by $- 1$ (remember to flip sign)

Thus, our answer is $a >- 2$. That means that, for any value of a that is greater than $- 2$, one solution will be to the left of $x = 1$, while the other will be to the right. Below are graphs using certain values for a:



8. **Solve $\sqrt{x - 4} + \sqrt{x + 1} + \sqrt{2x} - \sqrt{33 - x} > 4$. Find the sum of all integer solutions.**

Answer: 525

Solution: Please do not attempt to solve this algebraically. In this case, it is much easier to use an almost "brute force" method. We start by plugging in values to see what we get. Starting with 4 (because otherwise $\sqrt{x - 4}$ would be imaginary) we unfortunately cannot evaluate $\sqrt{x + 1}$. Noticing this problem, we move on to the next number 5, which has the same problem. However, once we reach 8, this problem no longer persists. We can evaluate the function at $x = 8$:

$$\sqrt{8 - 4} + \sqrt{8 + 1} + \sqrt{2(8)} - \sqrt{33 - 8} = \sqrt{4} + \sqrt{9} + \sqrt{16} - \sqrt{25} = 2 + 3 + 4 - 5 = 4$$

The function reaches a value of 4 at $x = 8$. We also know that the function just reached four from below four (because there are three square roots that increase as x increases and they would overpower the one that decreases) we can deduce that no values equal to or less than 8 satisfy the equation, but that values above 8 do. Noticing that that as x increases, the first three square roots increase and the fourth one decreases, we can conclude that as x increases, the sum of the square roots increases. Thus, the value stays greater than 4. However, notice that the greatest x value possible is 33 because otherwise $\sqrt{33 - x}$ would be imaginary. Therefore, our solutions are:

$$9, 10, 11, 12, \ldots, 33$$

Their sum (using the sum of an arithmetic sequence formula) is thus:

$$9 + 10 + 11 + 12 + \ldots + 33 = 25\frac{9+33}{2} = 25(21) = 525$$

Our answer is 525.

9. A coloring of the number in the set $\{2;\ 3;\ 4, \ldots, n\}$ in blue and red is called a "good" one if the product of every two (maybe equal) red numbers is blue, and the product of every two (maybe equal) blue numbers is red. Find the largest n for which "good" coloring exists.

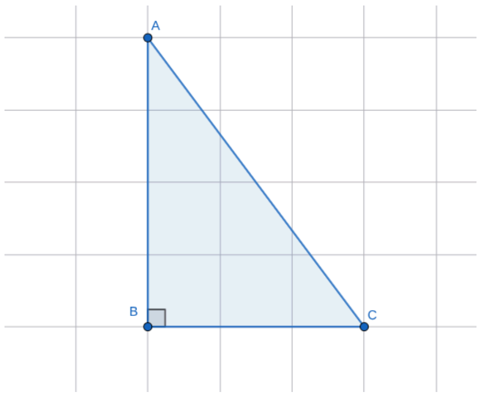
Answer: 31

Solution: I don't understand the problem or solution ATM. will update when I understand it.

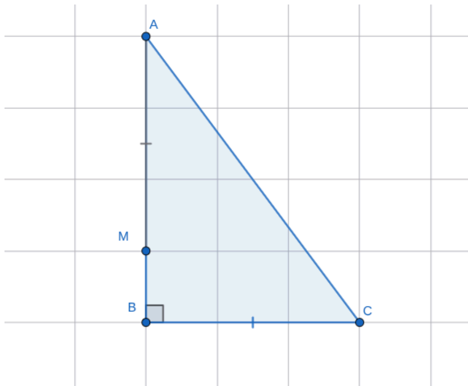
10. In a right triangle ABC, angle B is a right angle. Point M is located on line AB (M is not the midpoint of line AB), and it is known that $AM = BC$. Point N is located on line BC, and it is known that $CN = MB$. Find the measurement of an acute angle between line AN and line CM.

Answer:

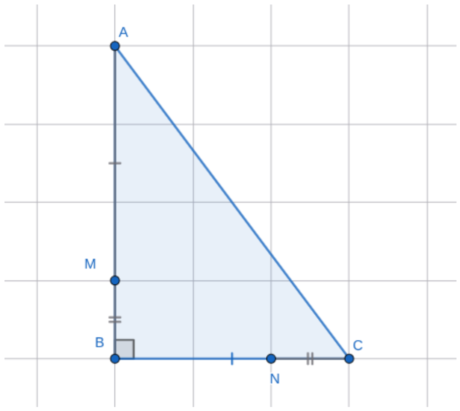
Solution: To begin with, make a sketch of the triangle ABC with a right angle B:



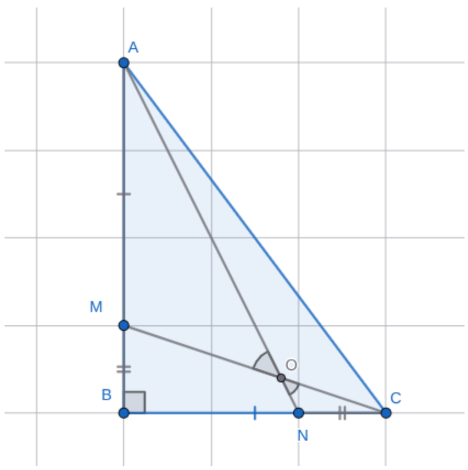
Now, put point M somewhere on segment AB (preferably not in the middle because we know it's not the midpoint of segment AB) and mark segment AM and segment BC congruent:



Now put a point N on segment BC such that segment CN and segment MB are congruent (and mark congruence):



To construct the acute angle between segment AN and segment CM, we create the segments AN and CM and then mark the angle between them (also keep in mind that I arbitrarily named their intersection point O to make referencing the angle easier in the future):



Keep in mind that I marked both angle MOA and angle NOC because they are both the acute angle the question wants (they are the same because they are vertical angles).

Competition Answer Key:

Central Jersey Mathematics League Contest

[Solutions]

September 17, 2025

1. If k, m, and n are positive numbers, and k is m percent of n, what percent of m is nk?

Solution: $k = \frac{m}{100} \cdot n$; $\frac{n\left(\frac{mn}{100}\right)}{m} \cdot 100 = n^2$. **Answer:** n^2.

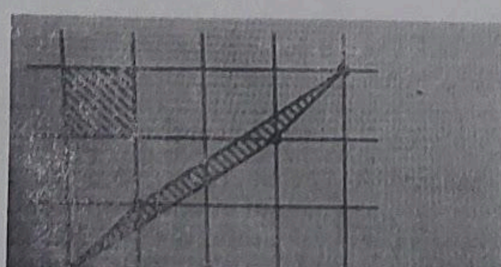
2. Find the maximum number of points of intersection among three circles and two lines in the same plane.

Solution: A pair of circles intersect in at most 2 points. For 3 circles, there are 3 pairs (thus, 6 points). A line can intersect a circle in at most 2 points (thus, 12 points). Two lines can intersect at 1 point. The total is $6 + 12 + 1 = 19$. **Answer: 19.**

3. There are 13 weights on the table, stacked in a row and ordered by their mass (the lightest is on the left, and the heaviest is on the right). It is known that the mass of each weight in grams represents an integer, the masses of two neighboring weights differ by no more than 5g, and the sum mass of all weights does not exceed 2025g. Find the maximum possible mass of the heaviest weight.

Solution: Let's assume that the heaviest weight has the mass equal m. The masses of the remaining weights have to be no less than $m - 5, m - 10, \dots, m - 60$, and their sum mass is no less than $13m - 390$. Then $13m - 390 \leq 2025$, and from here, $m \leq 185$. The set is: {185, 180, 175, 170, 165, 160, 155, 150, 145, 140, 135, 130, and 125}, where all numbers satisfy the initial conditions. **Answer: 185.**

4. Four vertices of a square grid where each square is 1by1, have been chosen to form a parallelogram. It turns out that neither the parallelogram's sides nor its interior contain any other vertices of the grid, and it is known that the parallelogram is not a rectangle. Find the area of the parallelogram.



Solution: In Fig. 1, use difference of areas of a large right triangle, a smaller right triangle, and a trapezoid to find an area of a half of the shaded figure, then multiply by 2: $\left(\frac{4 \cdot 3}{2} - \frac{3 \cdot 2}{2} - \frac{2+3}{2}\right) \cdot 2 = 12 - 6 - 5 = 1$. **Answer: 1 square unit.**
Note: The area is 1 for all qualifying parallelograms. Pick's Theorem (Formula) expresses the area of a polygon, all whose vertices are lattice points in a coordinate plane in terms of the number of lattice points inside the polygon and the number of lattice points on the sides of the polygon. The formula is: $A = I + \frac{1}{2}B - 1$, where A is the area of a polygon, I is the number of lattice points in the interior and B being the number of lattice points on the boundary. In our case, $A = 0 + \frac{4}{2} - 1 = 1$.
https://artofproblemsolving.com/wiki/index.php/Pick%27s_Theorem?srsltid=AfmBOopKgpEQOxLlqNpnT3jICMeK-bWVA9eArQDjfgyoLelXJg0vNA1w.

5. Emma took the word *GEOMETRY* and replaced each of its eight letters with a number according to the code $A = 1, B = 2, \dots, Z = 26$. Then she multiplied six of these numbers to get a cube of an integer. Find this integer.

Solution. First, let's recall the order of the letters in the alphabet to get the numbers Irene used to replace the letters: $G = 7, E = 5, O = 15, M = 13, E = 5, T = 20, R = 18, Y = 25$. Note that 7 and 13 are primes, and none of the other six numbers is a multiple of either 7 or 13. Emma had to exclude numbers 7 and 13 from multiplication, otherwise the result would be a multiple of 7 but not a multiple of 7^3 (or a multiple of 13 but not a multiple of 13^3), and therefore the result would not be a cube of an integer. Thus, Emma multiplied numbers 5, 15, 5, 20, 18, and 25, so she got $5 \cdot 15 \cdot 5 \cdot 20 \cdot 18 \cdot 25 = 5 \cdot (3 \cdot 5) \cdot 5 \cdot (2 \cdot 2 \cdot 5) \cdot (2 \cdot 3 \cdot 3) \cdot (5 \cdot$

$5) = 2^3 \cdot 3^3 \cdot 5^3 \cdot 5^3 = 150^3$. There is only one integer whose cube is 150^3, namely 150. **Answer: 150.**

6. Calculate the sum $1^2 + 2^2 - 3^2 - 4^2 + 5^2 + 6^2 - 7^2 - 8^2 + 9^2 + 10^2 - \dots - 2023^2 - 2024^2 + 2025^2$.

Solution: Notice, that $k^2 - (k+1)^2 - (k+2)^2 + (k+3)^2 = 4$ for every k. Thus, the desired sum is $1 + 506 \cdot 4 = 1 + 2024 = 2025$. **Answer: 2025.**

7. Find real a such that one solution of $x^2 - (3a+2)x + 2a - 1 = 0$ is greater than 1 and the other solution is less than 1?

Solution: Let's use a simple graphic interpretation. The graph of the function $x^2 - (3a+2)x + 2a - 1$ is a parabola, opening up. Under given condition, this parabola must intersect the x-axis, and the interval $[x_1, x_2]$ must include point $x = 1$: $x_1 < 1 < x_2$. The value of $x^2 - (3a+2)x + 2a - 1$ when $x = 1$ must be negative. Solving inequality $1^2 - (3a+2) \cdot 1 + 2a - 1 < 0$, we obtain the solution $a > -2$. **Answer: $a > -2$.**

8. Solve $\sqrt{x-4} + \sqrt{x+1} + \sqrt{2x} - \sqrt{33-x} > 4$. Find the sum of all integer solutions.

Solution: Rewrite the expression $\sqrt{x-4} + \sqrt{x+1} + \sqrt{2x} > \sqrt{33-x} + 4$. Define the domain of x:

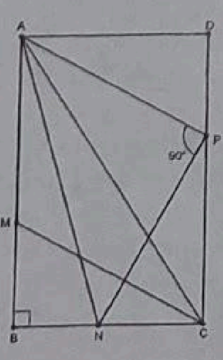
$$\begin{cases} x - 4 \geq 0 \\ x + 1 \geq 0 \\ 2x \geq 0 \\ 33 - x \geq 0, \end{cases}$$

From here, $x \in [4, 33]$. We are interested in the integers that satisfy the second (re-written) inequality above. Notice that the left side of this inequality is increasing, while the right side is decreasing. They share a common point: $x = 8$. When $x \in \{4, 5, 6, 7\}$, the left part is less than the right part; when $x \in \{9, 10, \dots, 33\}$, the left side is greater than the right side. Thus, calculate the sum $9 + 10 + \dots + 33 = \frac{9+33}{2} \cdot 25 = 525$. **Answer: 525.**

9. A coloring of the number in the set $\{2; 3; 4, \dots, n\}$ in blue and red is called a "good" one if the product of every two (maybe equal) red numbers is blue, and the product of every two (maybe equal) blue numbers is red. Find the largest n for which "good" coloring exists.

Solution. Estimation. Prove that the number 32 cannot be colored. Let assume that 2 is blue, $4 = 2 \cdot 2$ is red, $16 = 4 \cdot 4$ is blue, $32 = 2 \cdot 16$ is red. But 8 is neither red nor blue; if 8 is red, then there are three red numbers in the equality $8 \cdot 4 = 32$, and if 8 is blue, then there are three blue numbers in $8 \cdot 2 = 16$. So, 32 cannot be colored in any case. The largest n for which "good" coloring exists is 31. An example of a "good" coloring: let 2 and 3 be blue, the numbers from 4 to 15 red, and the numbers from 16 to 31 blue again. Another example: all the numbers having 1 or 4 prime factors (i. e. all primes as well as 16 and 24) are blue, and all the numbers having 2 or 3 prime factors are red. **Answer: 31.**

10. In a right triangle ABC, angle B is a right angle. Point M is located on $\overline{AB}$ (M is not the midpoint of $\overline{AB}$), and it is known that $AM = BC$. Point N is located on $\overline{BC}$, and it is known that $CN = MB$. Find the measurement of an acute angle between $\overline{AN}$ and $\overline{CM}$.



Solution: Draw a rectangle $ABCD$ around triangle ABC (Fig. 2) and choose point P on $\overline{CD}$ so that $\overline{AP}$ is parallel to $\overline{CM}$. Then, since $AM = BC = PC$, $PC = AD$, $PD = CN$, and the right triangles ADP and PCN are congruent. Then angle DAP is congruent to angle CPN by $CPCTC$. Thus, $m\angle APD + m\angle CPN = 90°$, and $m\angle APN = 90°$. Also, triangle APN is an isosceles right triangle, and $m\angle PAN = m\angle PNA = 45°$, and since $\overline{AP}$ and $\overline{CM}$ are parallel, $\overline{CM}$ is perpendicular to $\overline{NP}$, and the measure of an acute angle between $\overline{AN}$ and $\overline{CM}$ is $90° - 45° = 45°$. **Answer: 45 °.**